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# Branes, Rings and Matrix Models in Minimal (Super)string Theory

Nathan Seiberg[1] and David Shih[2]

[1]School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540 USA

[2]Department of Physics, Princeton University, Princeton, NJ 08544 USA

We study both bosonic and supersymmetric $(p, q)$ minimal models coupled to Liouville theory using the ground ring and the various branes of the theory. From the FZZT brane partition function, there emerges a unified, geometric description of all these theories in terms of an auxiliary Riemann surface $\mathcal{M}_{p,q}$ and the corresponding matrix model. In terms of this geometric description, both the FZZT and ZZ branes correspond to line integrals of a certain one-form on $\mathcal{M}_{p,q}$. Moreover, we argue that there are a finite number of distinct $(m, n)$ ZZ branes, and we show that these ZZ branes are located at the singularities of $\mathcal{M}_{p,q}$. Finally, we discuss the possibility that the bosonic and supersymmetric theories with $(p, q)$ odd and relatively prime are identical, as is suggested by the unified treatment of these models.

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## 1. Introduction and conclusions

In this work we will explore minimal string theories. These are simple examples of string theory with a small number of observables. Their simplicity makes them soluble and therefore they are interesting laboratories for string dynamics.

The worldsheet description of these minimal string theories is based on the $(p, q)$ minimal conformal field theories coupled to two-dimensional gravity (Liouville theory), or the $(p, q)$ minimal superconformal field theories coupled to two-dimensional supergravity (super-Liouville theory). Even though these two worldsheet descriptions appear different, we find that actually they are quite similar. In fact, our final answer depends essentially only on the values of $(p, q)$. This suggests a uniform presentation of all these theories which encompasses the two different worldsheet frameworks and extends them.

These theories were first solved using their description in terms of matrix models [1-8] (for reviews, see e.g. [9,10]). The matrix model realizes the important theme of open/closed string duality in the study of string theory. Recent advances in the study of Liouville theory [11-13] and its D-branes [14-16] has led to progress by [17-24] and others in making the connection between the matrix model and the worldsheet description more explicit. Much of the matrix model treatment of these theories has involved a deformation by the lowest dimension operator. In order to match with the worldsheet description based on Liouville theory, we should instead tune the background such that only the cosmological constant $\mu$ is turned on. In [25] such backgrounds were referred to as conformal backgrounds.

In the first part of this work, we will examine the simplest examples of minimal string theory, which are based on the bosonic $(p, q)$ minimal models coupled to two-dimensional gravity. These theories, which we review in section 2, exist for all relatively prime integers $(p, q)$. It is important that these theories have only a finite number of standard physical vertex operators $\mathcal{T}_{r,s}$ of ghost number one; we will refer to these as tachyons. They are constructed by "dressing" the minimal model primaries $\mathcal{O}_{r,s}$ with Liouville exponentials, subject to the condition that the combinations have conformal dimension one. They are labelled by integers $r = 1, \ldots, p-1$ and $s = 1, \ldots, q-1$, and they are subject to the identification $\mathcal{T}_{r,s} \sim \mathcal{T}_{p-r,q-s}$, which can be implemented by restricting to $qr > ps$.

In addition to the tachyons, there are infinitely many physical operators at other values of the ghost number [26]. Of special importance are the operators at ghost number zero. These form a ring [27] under multiplication by the operator product expansion modulo

BRST commutators. We will argue that the ring is generated by two operators $\widehat{\mathcal{O}}_{1,2}$ and $\widehat{\mathcal{O}}_{2,1}$. In terms of the operators

$$\widehat{x} = \frac{1}{2}\widehat{\mathcal{O}}_{1,2} \ , \quad \widehat{y} = \frac{1}{2}\widehat{\mathcal{O}}_{2,1} \ , \tag{1.1}$$

the other elements in the ring are

$$\widehat{\mathcal{O}}_{r,s} = U_{s-1}(\widehat{x})U_{r-1}(\widehat{y}) \tag{1.2}$$

where the $U$ are the Chebyshev polynomials of the second kind $U_{r-1}(\cos\theta) = \frac{\sin(r\theta)}{\sin\theta}$ and for simplicity we are setting here the cosmological constant $\mu = 1$. The ring has only $(p-1)(q-1)$ elements. The restriction on the values of $(r,s)$ is implemented by the ring relations

$$U_{q-1}(\widehat{x}) = 0 \ , \qquad U_{p-1}(\widehat{y}) = 0 \ . \tag{1.3}$$

The tachyons $\mathcal{T}_{r,s}$ form a module of the ring, which is simply understood by writing $\mathcal{T}_{r,s} = \widehat{\mathcal{O}}_{r,s}\mathcal{T}_{1,1}$. The restriction in the tachyon module $rq > ps$ implies that the module is not a faithful representation of the ring. Instead, there is an additional relation in the ring when acting on the tachyon module:

$$\big(U_{q-2}(\widehat{x}) - U_{p-2}(\widehat{y})\big)\mathcal{T}_{r,s} = 0 \ . \tag{1.4}$$

The ground ring will enable us to compute some simple correlation functions such as

$$\langle \mathcal{T}_{r_1,s_1}\mathcal{T}_{r_2,s_2}\mathcal{T}_{r_3,s_3}\rangle = N_{(r_1,s_1)(r_2,s_2)(r_3,s_3)}\langle \mathcal{T}_{1,1}\mathcal{T}_{1,1}\mathcal{T}_{1,1}\rangle \tag{1.5}$$

with $N_{(r_1,s_1)(r_2,s_2)(r_3,s_3)}$ the integer fusion rules of the minimal model.

In sections 3 and 4, we study the two types of branes in these theories, which are referred to as the FZZT and ZZ branes [14-16]. The former were previously explored in the context of matrix models as operators that create macroscopic loops [28,25,29,30]. The Liouville expressions for these loops [14,15] lead to more insight. The general Liouville expressions simplify in our case for two reasons. First, the Liouville coupling constant $b^2$ is rational, $b^2 = \frac{p}{q}$; and second, we are not interested in the generic Liouville operator but only in those which participate in the physical (i.e. BRST invariant) operators of the minimal string theory. It turns out that these branes are labelled by a continuous parameter $x$ which can be identified with the boundary cosmological constant $\mu_B$. This

parameter can be analytically continued, but it does not take values on the complex plane. Rather, it is defined on a Riemann surface $\mathcal{M}_{p,q}$ which is given by the equation

$$F(x,y) \equiv T_q(x) - T_p(y) = 0 \tag{1.6}$$

with $T_p(\cos\theta) = \cos(p\,\theta)$ a Chebyshev polynomial of the first kind. This Riemann surface has genus zero, but it has $(p-1)(q-1)/2$ singularities that can be thought of as pinched $A$-cycles of a higher-genus surface. In addition to (1.6), the singularities must satisfy

$$\begin{aligned} \partial_x F(x,y) &= q\, U_{q-1}(x) = 0 \\ \partial_y F(x,y) &= p\, U_{p-1}(y) = 0 \ . \end{aligned} \tag{1.7}$$

Rewriting the conditions (1.6) and (1.7) as

$$\begin{aligned} U_{q-1}(x) &= U_{p-1}(y) = 0 \\ U_{q-2}(x) &- U_{p-2}(y) = 0 \ , \end{aligned} \tag{1.8}$$

we immediately recognize the first line as the ring relations (1.3), and the second line as the relation in the tachyon module (1.4).

Deformations of the curve (1.6) correspond to the physical operators of the minimal string theory. Among them we find all the expected bulk physical operators, namely the tachyons, the ring elements and the physical operators at negative ghost number. There are also deformations of the curve that do not correspond to bulk physical operators. We interpret these to be open string operators.

A useful object is the one form $y\,dx$. Its integral on $\mathcal{M}_{p,q}$ from a fixed reference point (which we can take to be at $x \to \infty$) to the point $\mu_B$ is the FZZT disk amplitude with boundary cosmological constant $\mu_B$:

$$Z(\mu_B) = \int^{\mu_B} y\, dx \ . \tag{1.9}$$

We can also consider closed contour integrals of this one-form through the pinched cycles of $\mathcal{M}_{p,q}$. This leads to disk amplitudes for the ZZ branes:

$$Z_{m,n} = \oint_{B_{m,n}} y\, dx \ . \tag{1.10}$$

This integral can also be written as the difference between FZZT branes on the two sides of the singularity [24]. Such a relation between the two types of branes follows from the work of [16,31,32] and was made most explicit in [18].

The relations (1.8) which are satisfied at the singularities, together with the result (1.10), suggest the interpretation of the $(m, n)$ ZZ brane states as eigenstates of the ring generators $\widehat{x}$ and $\widehat{y}$, with eigenvalues $x_{mn}$ and $y_{mn}$ corresponding to the singularities of $\mathcal{M}_{p,q}$:

$$\widehat{x}|m,n\rangle_{\rm ZZ} = x_{mn}|m,n\rangle_{\rm ZZ} \ , \qquad \widehat{y}|m,n\rangle_{\rm ZZ} = y_{mn}|m,n\rangle_{\rm ZZ} \ . \tag{1.11}$$

We also find that these eigenvalues completely specify the BRST cohomology class of the ZZ brane. In other words, branes located at the same singularity of $\mathcal{M}_{p,q}$ differ by a BRST exact state, while branes located at different singularities are distinct physical states. Moreover, branes that do not correspond to singularities of our surface are themselves BRST exact. Thus there are as many distinct ZZ branes as there are singularities of $\mathcal{M}_{p,q}$ in $(p, q)$ minimal string theory.

A natural question is the physical interpretation of the uniformization parameter $\theta$ of our surface, defined by $x = \cos\theta$. The answer is given in appendix A, where we discuss the (nonlocal) Backlund transformation that maps the Liouville field $\phi$ to a free field $\tilde{\phi}$. We will see that $\theta$ can be identified with the Backlund field, which satisfies Dirichlet boundary conditions.

In the second part of our paper we will study the minimal superstring theories, which are obtained by coupling $(p, q)$ superminimal models to supergravity. These theories fall in two classes. The odd models exist for $p$ and $q$ odd and relatively prime. They (spontaneously) break worldsheet supersymmetry. The even models exist for $p$ and $q$ even, $p/2$ and $q/2$ relatively prime and $(p - q)/2$ odd. Our discussion parallels that in the bosonic string and the results are very similar. There are however a few new elements.

The first difference from the bosonic system is the option of using the 0A or 0B GSO projection. Most of our discussion will focus on the 0B theory. In either theory, there is a global $\mathbb{Z}_2$ symmetry $(-1)^{F_L}$, where $F_L$ is the left-moving spacetime fermion number. This symmetry multiplies all the RR operators by $-1$ and is broken when background RR fields are turned on. Orbifolding the 0B theory by $(-1)^{F_L}$ leads to the 0A theory and vice versa.

A second important difference relative to the bosonic system is that here the cosmological constant $\mu$ can be either positive or negative, and the results depend on its sign

$$\zeta = \mathrm{sign}(\mu) \ . \tag{1.12}$$

This sign can be changed by performing a $\mathbb{Z}_2$ R-transformation in the super-Liouville part of the theory. It acts there as $(-1)^{f_L}$ with $f_L$ the left moving worldsheet fermion number.

Since this is an R-transformation, it does not commute with the worldsheet supercharge. In order to commute with the BRST charge such an operation must act on the total supercharge including the matter part. This operation is usually not a symmetry of the theory. In particular it reverses the sign of the GSO projection in the Ramond sector. Therefore, our answers will in general depend on $\zeta$.

In section 5 we discuss the tachyons, the ground ring and the correlation functions. The results are essentially identical to those in the bosonic string except that we should use the appropriate values of $(p,q)$, and the ring relation in the tachyon module (1.4) depends on $\zeta$

$$\big(U_{q-2}(\widehat{x}) - \zeta U_{p-2}(\widehat{y})\big)\mathcal{T}_{r,s} = 0\ . \tag{1.13}$$

In sections 6 and 7 we consider the supersymmetric FZZT and ZZ branes. Here we find a third element which is not present in the bosonic system. Now there are two kinds of branes labelled by a parameter $\eta = \pm 1$; this parameter determines the combination of left and right moving supercharges that annihilates the brane boundary state: $(G + i\eta\tilde{G})|B\rangle = 0$. We must also include another label $\xi = \pm 1$ which multiplies the Ramond component of the boundary state. It is associated with the $\mathbb{Z}_2$ symmetry $(-1)^{F_L}$. Changing the sign of $\xi$ maps a brane to its antibrane.

Most of these new elements do not affect the odd models. The answers are independent of $\zeta$ and $\eta$, and we again find the Riemann surface $\mathcal{M}_{p,q}$ given by the curve $T_q(x) - T_p(y) = 0$. The characterization of the FZZT and ZZ branes as contour integrals of $y\,dx$ is identical to that in the bosonic string.

The even models are richer. Here we find two Riemann surfaces $\mathcal{M}^{\widehat{\eta}}_{p,q}$ depending on the sign of

$$\widehat{\eta} = \zeta\eta\ . \tag{1.14}$$

These surfaces are described by the curves

$$F_{\widehat{\eta}}(x,y) = T_q(x) - \widehat{\eta}\,T_p(y) = 0\ . \tag{1.15}$$

The discussion of the surface $\mathcal{M}^{-}_{p,q}$ is similar to that of the bosonic models. The surface $\mathcal{M}^{+}_{p,q}$ is more special, because it splits into two separate subsurfaces $T_{\frac{p}{2}}(y) = \xi T_{\frac{q}{2}}(x)$ which touch each other at singular points. The two subsurfaces are interpreted as associated with the choice of the Ramond "charge" $\xi = \pm 1$. The two kinds of FZZT branes which are labelled by $\xi$ correspond to line integrals from infinity in the two sub-surfaces. ZZ branes are again given by contour integrals. These can be either closed contours which

pass through the pinched singularities in each subsurface, or they can be associated with contour integrals from infinity in one subsurface through a singularity which connects the two subsurfaces to infinity in the other subsurface.

It is surprising that our results depend essentially only on $p$ and $q$. The main difference between the bosonic models and the supersymmetric models is in the allowed values of $(p,q)$. Odd $p$ and $q$ which are relatively prime occur in both the bosonic and the supersymmetric models. This suggests that these two models might in fact be the same. This suggestion is further motivated by the fact that the two theories have the same KPZ scalings [24,33]. Indeed, all our results for these models (ground ring, sphere three point function, FZZT and ZZ branes) are virtually identical. We will discuss the evidence for the equivalence of these models in section 8.

Our work goes a long way to deriving the matrix model starting from the worldsheet formulation of the theory. We will discuss the comparison with the matrix model in section 9. Our Riemann surface $\mathcal{M}_{p,q}$ occurs naturally in the matrix model and determines the eigenvalue distribution. Of all the possible matrix model descriptions of the minimal string theories, the closest to our approach are Kostov's loop gas model [30-37] and the two matrix model [38], in which expressions related to ours were derived. For the theories with $p = 2$, we also have a description in terms of a one-matrix model. Some more detailed aspects of the comparison to the one-matrix model are worked out in appendix B.

The matrix model also allows us to explore other values of $(p,q)$ which are not on the list of (super)minimal models. For example, the theories with $(p,q) = (2, 2k+2)$ were interpreted in [24] as a minimal string theory with background RR fields. It is natural to conjecture that all values of $(p,q)$ correspond to some minimal string theory or deformations thereof. This generalizes the worldsheet constructions based on (super)minimal models and (super)Liouville theory.

The Riemann surface which is central in our discussion is closely related to the target space of the eigenvalues of the matrix model. However, it should be stressed that neither the eigenvalue direction nor the Riemann surface are the target space of the minimal string theory itself. Instead, the target space is the Liouville direction $\phi$, which is related to the eigenvalue space through a nonlocal transform [29]. This distinction between $\phi$ and the coordinates of the Riemann surface is underscored by the fact that $x$ and $y$ on the Riemann surface are related to composite operators in the worldsheet theory (the ground ring generators), rather than to the worldsheet operator $\phi$.

Our discussion is limited to the planar limit, where the worldsheet topology is a sphere or a disk (with punctures). It would be interesting to extend it to the full quantum string theory. It is likely that the work of [39,40] is a useful starting point of this discussion.

A crucial open problem is the nonperturbative stability of these theories. For instance, some of the bosonic $(p, q)$ theories are known to be stable while others are known to be unstable. In fact, as we will discuss in section 9, the matrix model of all these models have a small instability toward the tunnelling of a small number of eigenvalues. The nonperturbative status of models based on generic values of $(p, q)$ remains to be understood.

After the completion of this work an interesting paper [41] came out which overlaps with some of our results and suggests an extension to the quantum string theory.

## 2. Bosonic minimal string theory

### *2.1. Preliminaries*

We start by summarizing some relevant aspects of the $(p, q)$ minimal models and Liouville theory that we will need for our analysis, at the same time establishing our notations and conventions. For the bosonic theories, we will take $\alpha' = 1$. The $(p, q)$ minimal models exist for all $p, q \geq 2$ coprime. Our convention throughout will be $p < q$. The central charge of these theories is given by

$$c = 1 - \frac{6(p-q)^2}{p\,q} \tag{2.1}$$

The $(p, q)$ minimal model has a total of $N_{p,q} = (p-1)(q-1)/2$ primary operators $\mathcal{O}_{r,s}$ labelled by two integers $r$ and $s$, with $r = 1, \ldots, p-1$ and $s = 1, \ldots, q-1$. They satisfy the reflection relation $\mathcal{O}_{p-r,q-s} \equiv \mathcal{O}_{r,s}$ and have dimensions

$$\Delta(\mathcal{O}_{r,s}) = \overline{\Delta}(\mathcal{O}_{r,s}) = \frac{(rq - sp)^2 - (p-q)^2}{4p\,q} \ . \tag{2.2}$$

In general the operator $\mathcal{O}_{r,s}$ contains two primitive null vectors among its conformal descendants at levels $rs$ and $(p-r)(q-s)$. Because they are degenerate, primary operators must satisfy the fusion rules

$$\begin{aligned} \mathcal{O}_{r_1,s_1}\mathcal{O}_{r_2,s_2} = \sum [\mathcal{O}_{r,s}] \\ r = |r_1 - r_2| + 1, |r_1 - r_2| + 3, \ldots, \\ \min(r_1 + r_2 - 1, 2p - 1 - r_1 - r_2) \\ s = |s_1 - s_2| + 1, |s_1 - s_2| + 3, \ldots, \\ \min(s_1 + s_2 - 1, 2q - 1 - s_1 - s_2) \end{aligned} \tag{2.3}$$

The fusion of any operator with $\mathcal{O}_{1,2}$ and $\mathcal{O}_{2,1}$ is especially simple:

$$\begin{aligned}\mathcal{O}_{1,2}\mathcal{O}_{r,s} &= [\mathcal{O}_{r,s+1}] + [\mathcal{O}_{r,s-1}] \\ \mathcal{O}_{2,1}\mathcal{O}_{r,s} &= [\mathcal{O}_{r+1,s}] + [\mathcal{O}_{r-1,s}] \ .\end{aligned} \tag{2.4}$$

Thus in a sense, $\mathcal{O}_{1,2}$ and $\mathcal{O}_{2,1}$ generate all the primary operators of the minimal model. In particular, $\mathcal{O}_{1,2}$ generates all primaries of the form $\mathcal{O}_{1,s}$ and $\mathcal{O}_{2,1}$ generates all primaries of the form $\mathcal{O}_{r,1}$. The fusion of two such operators is very simple: $\mathcal{O}_{1,s}\mathcal{O}_{r,1} = [\mathcal{O}_{r,s}]$.

The central charge of Liouville field theory is

$$c = 1 + 6Q^2 = 1 + 6\left(b + \frac{1}{b}\right)^2 \tag{2.5}$$

where $Q = b + \frac{1}{b}$ is the background charge. The basic primary operators of the Liouville theory are the vertex operators $V_\alpha = e^{2\alpha\phi}$ with dimension

$$\Delta(\alpha) = \overline{\Delta}(\alpha) = \alpha(Q - \alpha) \ . \tag{2.6}$$

Of special interest are the primaries in degenerate Virasoro representations

$$V_{\alpha_{r,s}} = e^{2\alpha_{r,s}\phi}, \quad 2\alpha_{r,s} = \frac{1}{b}(1-r) + b(1-s) \ . \tag{2.7}$$

For generic $b$, these primaries have exactly one singular vector at level $rs$. Therefore, their irreducible character is given by:

$$\chi_{r,s}(q) = \frac{q^{\Delta(\alpha_{r,s})-(c-1)/24}}{\eta(q)}(1 - q^{rs}) \tag{2.8}$$

The degenerate primaries also satisfy the analogue of the minimal model fusion rule (2.4) [11,13]:

$$\begin{aligned}V_{-\frac{b}{2}}V_\alpha &= [V_{\alpha-\frac{b}{2}}] + C(\alpha)[V_{\alpha+\frac{b}{2}}] \\ C(\alpha) &= -\mu\frac{\pi\gamma(2b\alpha - 1 - b^2)}{\gamma(-b^2)\gamma(2b\alpha)}\end{aligned} \tag{2.9}$$

with a similar expression for $V_{-\frac{1}{2b}}$:

$$\begin{aligned}V_{-\frac{1}{2b}}V_\alpha &= [V_{\alpha-\frac{1}{2b}}] + \tilde{C}(\alpha)[V_{\alpha+\frac{1}{2b}}] \\ \tilde{C}(\alpha) &= -\tilde{\mu}\frac{\pi\gamma(2\alpha/b - 1 - 1/b^2)}{\gamma(-1/b^2)\gamma(2\alpha/b)} \ .\end{aligned} \tag{2.10}$$

Here $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. Notice that the second OPE may be obtained from the first by taking $b \to 1/b$ and also $\mu \to \tilde{\mu}$. The quantity $\tilde{\mu}$ is the *dual cosmological constant*, which is related to $\mu$ via $\pi\tilde{\mu}\gamma(1/b^2) = (\pi\mu\gamma(b^2))^{1/b^2}$. From this point onwards, we will find it convenient to rescale $\mu$ and $\tilde{\mu}$ so that

$$\tilde{\mu} = \mu^{1/b^2} \ . \tag{2.11}$$

This will simply many of our later expressions.

In order to construct a minimal string theory, we must couple the $(p,q)$ minimal model to Liouville. Demanding the correct total central charge implies that the Liouville theory must have

$$b = \sqrt{\frac{p}{q}} \ . \tag{2.12}$$

We will see throughout this work that taking $b^2$ to be rational and restricting only to the BRST cohomology of the full string theory gives rise to many simplifications, and also to a number of subtleties. One such a simplification is that not all Liouville primaries labelled by $\alpha$ correspond to physical (i.e. BRST invariant) vertex operators of minimal string theory. For instance, physical vertex operators may be formed by first "dressing" an operator $\mathcal{O}_{r,s}$ from the matter theory with a Liouville primary $V_{\beta_{r,s}}$ such that the combination has dimension $(1,1)$, and then multiplying them with the ghosts $c\bar{c}$. We will refer to such operators as "tachyons." Requiring the sum of (2.2) and (2.6) to be 1 gives the formula for the Liouville dressing of $\mathcal{O}_{r,s}$:

$$\begin{aligned} \mathcal{T}_{r,s} &= c\bar{c}\mathcal{O}_{r,s}V_{\beta_{r,s}} \\ 2\beta_{r,s} &= \frac{p+q-rq+sp}{\sqrt{pq}}, \quad rq - sp \geq 0 \ . \end{aligned} \tag{2.13}$$

Note that in solving the quadratic equation for $\beta_{r,s}$, we have chosen the branch of the square root so that $\beta_{r,s} < Q/2$ [42].

An important subtlety that arises at rational $b^2$ has to do with the irreducible character of the degenerate primaries. The formula (2.8) that we gave above for generic $b$ must now be modified for a number of reasons.[1]

[1] We thank B. Lian and G. Zuckerman for helpful discussions about the structure of these representations.

1. Different $(r, s)$ can now lead to the same degenerate primary, and therefore labelling the representations with $(r, s)$ is redundant. It will sometimes be convenient to remove this redundancy by defining

$$N(t, m, n) \equiv |\, tpq + mq + np \,| \tag{2.14}$$

and labelling each representation by $(t, m, n)$ satisfying

$$0 < m \leq p \ , \qquad 0 < n \leq q \ , \qquad t \geq 0 \tag{2.15}$$

such that the degenerate primary has dimension

$$\Delta(t, m, n) = \frac{(p+q)^2 - N(t, m, n)^2}{4pq} \tag{2.16}$$

Then each $(t, m, n)$ satisfying (2.15) corresponds to a unique degenerate representation and vice versa.

2. The Verma module of a degenerate primary can now have more than one singular vector. For instance, we can write

$$N(t, m, n) = ((t-j)p + m)q + (jq + n)p \tag{2.17}$$

for $j = 0, \ldots, t$, and by continuity in $b$ we expect the Verma module of $(t, m, n)$ to contain singular vectors at levels $((t-j)p + m)(jq + n)$, with dimensions

$$\Delta = \frac{(p+q)^2 - N(t-2j, m, -n)^2}{4pq} \tag{2.18}$$

3. A related subtlety is that a singular vector can itself be degenerate. For example, the singular vectors discussed above are degenerate as long as $t - 2j \neq 0$. In general, this will lead to a complicated structure of nested Verma submodules contained within the original Verma module of the degenerate primary. A similar structure is seen, for instance, in the minimal models at $c < 1$. An important distinction is that here there are only finitely many singular vectors.

Taking into account these subtleties leads to a formula for the character slightly more complicated than the naive formula (2.8). The answer is [43,44]:

$$\widehat{\chi}_{t,m,n}(q) = \frac{1}{\eta(q)} \sum_{j=0}^{t} \left( q^{-N(t-2j,m,n)^2/4pq} - q^{-N(t-2j,m,-n)^2/4pq} \right) \tag{2.19}$$

Notice that we can also write this as a sum over naive characters (2.8):

$$\widehat{\chi}_{t,m,n}(q) = \sum_{j=0}^{[\frac{t}{2}]} \chi_{(t-2j)p+m,n} - \sum_{j=0}^{[\frac{t-1}{2}]} \chi_{(t-2j)p-m,n} \tag{2.20}$$

For $t = 0$, the formula reduces to (2.8), i.e. $\widehat{\chi}_{t=0,m,n} = \chi_{m,n}$. These results will be important when we come to discuss the ZZ boundary states of minimal string theory in section 3.

*2.2. The ground ring of minimal string theory*

The other important collection of BRST invariant operators in minimal string theory is the ground ring of the theory. The ground ring consists of all dimension 0, ghost number 0, primary operators in the BRST cohomology of the theory [26], and it was first studied for the $c = 1$ bosonic string in [27]. Ring multiplication is provided by the OPE modulo BRST commutators. As in the matter theory, elements $\widehat{\mathcal{O}}_{r,s}$ of the ground ring are labelled by two integers $r$ and $s$, $r = 1, \ldots, p-1$ and $s = 1, \ldots, q-1$. In contrast to the matter primaries, however, $\widehat{\mathcal{O}}_{r,s}$ and $\widehat{\mathcal{O}}_{p-r,q-s}$ are distinct operators. Thus the ground ring has $(p-1)(q-1)$ elements, twice as many as the matter theory.

The construction of the ground ring starts by considering the combination $\mathcal{O}_{r,s}V_{\alpha_{r,s}}$ of a matter primary and a corresponding degenerate Liouville primary. Using (2.2) and (2.6), it is easy to see that this combination has dimension $1-rs$. Acting on $\mathcal{O}_{r,s}V_{\alpha_{r,s}}$ with a certain combination of level $rs-1$ raising operators then gives the ground ring operator

$$\widehat{\mathcal{O}}_{r,s} = \mathcal{L}_{r,s} \cdot \mathcal{O}_{r,s}V_{\alpha_{r,s}}, \qquad 2\alpha_{r,s} = \frac{p+q-rq-sp}{\sqrt{pq}} \ . \tag{2.21}$$

From the construction, it is clear that $\widehat{\mathcal{O}}_{r,s}$ has Liouville momentum $2\alpha_{r,s}$. When $b$ and $1/b$ are incommensurate (which is the case for the $(p,q)$ minimal models), (2.21) then implies that there is a unique ground ring operator at a given Liouville momentum $2\alpha_{r,s}$ for $r = 1, \ldots, p-1$ and $s = 1, \ldots, q-1$. It follows that the multiplication table for the ground ring can be derived from kinematics alone. When $\mu = 0$, Liouville momentum is conserved in the OPE, and therefore one must have

$$\widehat{\mathcal{O}}_{r,s} = \widehat{\mathcal{O}}_{1,2}^{s-1}\widehat{\mathcal{O}}_{2,1}^{r-1} \ . \tag{2.22}$$

Thus the ground ring is generated by two elements, $\widehat{\mathcal{O}}_{1,2}$ and $\widehat{\mathcal{O}}_{2,1}$. Moreover, the fact that the ring has finitely many elements leads to non-trivial relations for the ring generators:

$$\begin{aligned}\widehat{\mathcal{O}}_{1,2}^{q-1} &= 0 \\ \widehat{\mathcal{O}}_{2,1}^{p-1} &= 0\ .\end{aligned} \tag{2.23}$$

Both (2.22) and (2.23) (and all the ground ring equations that follow) are understood to be true modulo BRST commutators.

Using the $\mu$-deformed Liouville OPEs (2.9), (2.10) it is easy to see how ring multiplication and the ring relations are altered at $\mu \neq 0$. It will be convenient to define the dimensionless combinations

$$\widehat{x} = \frac{1}{2\sqrt{\mu}}\widehat{\mathcal{O}}_{1,2}, \qquad \widehat{y} = \frac{1}{2\sqrt{\tilde{\mu}}}\widehat{\mathcal{O}}_{2,1}\ . \tag{2.24}$$

Let us start by analyzing the operators $\widehat{\mathcal{O}}_{r,1}$. It is clear that $\widehat{\mathcal{O}}_{r,1} = \mathcal{P}_{r-1}(\widehat{y})$ is a polynomial of degree $r-1$ in the generator $\widehat{y}$. These polynomials are constrained by the fusion rules

$$\begin{aligned}\mathcal{P}_{r-1}(\widehat{y})\mathcal{P}_{l-1}(\widehat{y}) &= \sum a_{r,l,k}\mathcal{P}_{k-1}(\widehat{y}) \\ k &= |r-l|+1, |r-l|+3, \dots, \\ &\qquad \min(r+l-1, 2p-1-r-l)\end{aligned} \tag{2.25}$$

The restrictions on the sum in (2.25) determine most of the coefficients in $\mathcal{P}_{r-1}$ even without using the known values of the operator product coefficients ((2.9) and the similar coefficient in the minimal model).[2] The remaining coefficients can be computed using the OPE in the minimal model and Liouville, but we will not do that here. Instead we will simply state the answer. We claim that

$$\widehat{\mathcal{O}}_{r,1} = \mu^{\frac{q(r-1)}{2p}} U_{r-1}(\widehat{y}) \tag{2.26}$$

where the $U$ are Chebyshev polynomials of the second kind. We will postpone the full justification of our claim until we come to discuss the ZZ brane one-point functions in

[2] For example, all the coefficients in $\mathcal{P}_r$ and all the coefficients $a_{r,l,k}$ in (2.25) are determined in terms of $a_{1,r,r-1}$ which appears in $\mathcal{P}_1(\widehat{y})\mathcal{P}_r(\widehat{y}) = \mathcal{P}_{r+1}(\widehat{y}) + a_{1,r,r-1}\mathcal{P}_{r+1}(\widehat{y})$. The $p$-dependence of the truncation of the sum in (2.25) leads to the ring relation $\mathcal{P}_{p-1}(\widehat{y}) = 0$, which generalizes (2.23) to nonzero $\mu$, and leads to relations among the coefficients $a_{1,r,r-1}$.

section 3.1. Also, the computation of the tachyon three-point functions below will provide a non-trivial check of (2.26).

For now, let us simply show that our ansatz is consistent with (2.25). This is a result of the following trigonometric identity for the multiplication of the Chebyshev polynomials $U$:

$$\begin{gathered} U_{r-1}(\widehat{y})U_{l-1}(\widehat{y}) = \sum_k U_{k-1}(\widehat{y}) \\ k = |r-l|+1, |r-l|+3, \ldots, r+l-1 \ . \end{gathered} \tag{2.27}$$

The fact that the Chebyshev polynomials can be expressed as $SU(2)$ characters

$$U_{r-1}(\cos\theta) = \frac{\sin(r\theta)}{\sin\theta} = \mathrm{Tr}_{j=\frac{r-1}{2}} e^{2i\theta J_3} \ . \tag{2.28}$$

underlies the identity (2.27). This identity is almost of the form (2.25). The $p$-dependence in (2.25) is implemented by the ring relation

$$U_{p-1}(\widehat{y}) = 0 \ . \tag{2.29}$$

(This is a standard fact in the representation theory of $\widehat{SU(2)}$.) This shows that the expression (2.26) with the relation (2.29) satisfies (2.25) with all nonzero $a_{r,l,k}$ equal to one.

It is trivial to extend these results to the operators $\widehat{\mathcal{O}}_{1,s}$ which are generated by $\widehat{x}$. Finally, using $\widehat{\mathcal{O}}_{r,s} = \widehat{\mathcal{O}}_{r,1}\widehat{\mathcal{O}}_{1,s}$ we derive the expressions for the ring elements

$$\widehat{\mathcal{O}}_{r,s} = \mu^{\frac{q(r-1)+p(s-1)}{2p}} U_{s-1}(\widehat{x})U_{r-1}(\widehat{y}) \tag{2.30}$$

and the relations

$$\begin{gathered} U_{q-1}(\widehat{x}) = 0 \\ U_{p-1}(\widehat{y}) = 0 \ . \end{gathered} \tag{2.31}$$

Having found the ring multiplication, we can use it to analyze the tachyon operators (2.13). Ghost number conservation implies that the tachyons are a module of the ring [45]. Using our explicit realization in terms of the generators (2.30) it is clear that

$$\mathcal{T}_{r,s} = \mu^{1-s}\widehat{\mathcal{O}}_{r,s}\mathcal{T}_{1,1} = \mu^{\frac{q(r-1)-p(s-1)}{2p}} U_{s-1}\big(\widehat{x}\big)U_{r-1}\big(\widehat{y}\big)\mathcal{T}_{1,1} \ . \tag{2.32}$$

Using this expression it is easy to act on $\mathcal{T}_{r,s}$ with any ring operator. This can be done by writing the ring operator and the tachyon in terms of the generators as in (2.30) and

(2.32) and then simply multiplying the polynomials in the generators subject to the ring relations (2.31).

It is clear, however, that this cannot be the whole story. There are $(p-1)(q-1)$ different ring elements $\widehat{\mathcal{O}}_{r,s}$, but there are only $(p-1)(q-1)/2$ tachyons $\mathcal{T}_{r,s}$ because they are subject to the identification

$$\mathcal{T}_{p-r,q-s} = \mu^{\frac{ps-qr}{p}} \mathcal{T}_{r,s} \ . \tag{2.33}$$

This means that in addition to the two ring relations (2.31) there must be more relations which are satisfied only in the tachyon module; i.e. it is not a faithful representation of the ring. It turns out that one should impose

$$\big(U_{q-2}(\widehat{x}) - U_{p-2}(\widehat{y})\big)\mathcal{T}_{r,s} = 0 \ . \tag{2.34}$$

It is easy to show, using trigonometric/Chebyshev identities, that this relation guarantees the necessary identifications [46] (see also [47,48]). Note that equation (2.34) can also be written using the ring relations (2.31) in terms of the Chebyshev polynomials of the first kind $T_p(x) = \cos(p\,\theta)$ as

$$\big(T_q(\widehat{x}) - T_p(\widehat{y})\big)\mathcal{T}_{r,s} = 0 \ . \tag{2.35}$$

This expression will be useful in later sections. We should also point out that the effect of the relations (2.31) and (2.34) on ring multiplication is to truncate it to precisely the fusion rules (2.3) of the minimal model.

Using this understanding we can constrain the correlation functions of these operators. The simplest correlation functions involve three tachyons and any number of ring elements on the sphere. Because of the conformal Killing vectors on the sphere, this calculation does not involve any moduli integration. It is given by

$$\langle \mathcal{T}_{r_1,s_1}\mathcal{T}_{r_2,s_2}\mathcal{T}_{r_3,s_3}\prod_{i\geq 4}\widehat{\mathcal{O}}_{r_i,s_i}\rangle = \mu^{3-s_1-s_2-s_3}\langle \mathcal{T}_{1,1}\mathcal{T}_{1,1}\mathcal{T}_{1,1}\prod_{i\geq 1}\widehat{\mathcal{O}}_{r_i,s_i}\rangle \ . \tag{2.36}$$

The product of ring elements can be recursively simplified using the ring relations (2.31) and (2.34). This leads to a linear combination of ring elements, of which only $\widehat{\mathcal{O}}_{1,1}$ survives in the expectation value (2.36). As an example, consider the three-point function:

$$\begin{aligned}\langle \mathcal{T}_{r_1,s_1}\mathcal{T}_{r_2,s_2}\mathcal{T}_{r_3,s_3}\rangle &= \mu^{3-s_1-s_2-s_3}\langle \mathcal{T}_{1,1}\mathcal{T}_{1,1}\mathcal{T}_{1,1}\widehat{\mathcal{O}}_{r_1,s_1}\widehat{\mathcal{O}}_{r_2,s_2}\widehat{\mathcal{O}}_{r_3,s_3}\rangle \\ &= N_{(r_1,s_1)(r_2,s_2)(r_3,s_3)}\mu^{\kappa}\end{aligned} \tag{2.37}$$

where $N_{(r_1,s_1)(r_2,s_2)(r_3,s_3)} \in \{0,1\}$ represent the integer fusion rules of the conformal field theory and $\kappa$ is the KPZ exponent of the correlation function:

$$\kappa = \frac{Q - \sum_i \beta_{r_i,s_i}}{b} \ . \tag{2.38}$$

Note that in calculating the three-point function, we have normalized $\langle \mathcal{T}_{1,1}^3 \rangle = \mu^{\frac{Q}{b}-3}$.

The surprising result that the three-point functions are given simply by the minimal model fusion rules was noticed many years ago using different methods [49-51]. The agreement between our calculation and the results in [49-51] serves as a check of our ansatz (2.30) for the $\mu$-deformed ring multiplication. (We will also give an independent derivation of (2.30) in section 3.1 using the ZZ branes.) From our current perspective, the simplicity of the tachyon three-point functions follows from the simplicity of the expressions for the ring elements (2.30) and the relations (2.31), (2.34). At a superficial level we can view the minimal string theory as a topological field theory based on the chiral algebra $Virasoro/Virasoro$. Its physical operators are the primaries of the conformal field theory and its three point functions are the fusion rule coefficients.

For correlation functions involving fewer than three tachyons we simply insert the necessary power of the cosmological constant $\mathcal{T}_{1,1}$ in order to have three tachyons. Then we integrate with respect to $\mu$ to find the desired two- or one-point functions.

It would be nice to generalize this method to four and higher-point correlation functions. For such correlators, there are potential complications involving contact terms between the integrated vertex operators.

## 3. FZZT and ZZ branes of minimal string theory

### *3.1. Boundary states and one-point functions*

In this section, we will study in detail the FZZT and ZZ branes of Liouville theory coupled to the bosonic $(p,q)$ minimal models. Let us first review what is known about the FZZT and ZZ boundary states in these theories. To make the former, we must tensor together an FZZT boundary state in Liouville and a Cardy state from the matter. This gives the boundary state [14,15,52]:

$$|\sigma; k, l\rangle = \sum_{k',l'} \int_0^\infty dP \, \cos(2\pi P\sigma) \frac{\Psi^*(P) S(k,l;k',l')}{\sqrt{S(1,1;k',l')}} |P\rangle\rangle_L |k',l'\rangle\rangle_M \ . \tag{3.1}$$

Here $(k,l)$ labels the matter Cardy state associated to the minimal model primary $\mathcal{O}_{k,l}$ and $|p\rangle\rangle_L$ and $|k',l'\rangle\rangle_M$ are Liouville and matter Ishibashi states, respectively. The Liouville and matter wavefunctions are $\cos(2\pi P\sigma)\Psi(P)$ and $S(k,l;k',l')$, where

$$\begin{aligned}\Psi(P) &= \mu^{-\frac{iP}{b}}\frac{\Gamma\left(1+\frac{2iP}{b}\right)\Gamma(1+2iPb)}{i\pi P}\\ S(k,l;k',l') &= (-1)^{kl'+k'l}\sin(\frac{\pi p l l'}{q})\sin(\frac{\pi q k k'}{p})\ .\end{aligned}\tag{3.2}$$

The matter wavefunction is essentially the modular $S$-matrix of the minimal model. Note that we have separated out the $\sigma$ dependent part of the Liouville wavefunction for later convenience. Finally, the parameter $\sigma$ is related to the boundary cosmological constant $\mu_B$ via[3]

$$\frac{\mu_B}{\sqrt{\mu}} = \cosh\pi b\sigma\ .\tag{3.3}$$

By thinking of the FZZT states as Liouville analogues of Cardy states, one also finds that the state labelled by $\sigma$ is associated to the non-degenerate Liouville primary with $2\alpha = Q + i\sigma$ [16].

Using the expression (3.1) for the FZZT boundary state, we can easily calculate the one-point functions of physical operators on the disk with the FZZT boundary condition. Let us start with the physical tachyon operators $\mathcal{T}_{r,s} = \mathcal{O}_{r,s}e^{2\beta_{r,s}\phi}$ with $\beta_{r,s}$ defined in (2.13). Using $P = i(Q/2-\beta_{r,s})$ in (3.1), we find

$$\langle\mathcal{T}_{r,s}|\,\sigma;k,l\rangle = A_{r,s}(-1)^{ks+lr}\cosh\left(\frac{\pi(qr-ps)\sigma}{\sqrt{pq}}\right)\sin(\frac{\pi q k r}{p})\sin(\frac{\pi p l s}{q})\tag{3.4}$$

where the $(\sigma,k,l)$ independent normalization factor $A_{r,s}$ will be irrelevant for our purposes.

We can similarly compute the one-point functions for the ground ring elements $\widehat{\mathcal{O}}_{r,s}$. As discussed in section 2, these take the general form

$$\widehat{\mathcal{O}}_{r,s} = \mathcal{L}_{r,s}\cdot\mathcal{O}_{r,s}e^{2\alpha_{r,s}\phi}\tag{3.5}$$

where $\mathcal{L}_{r,s}$ denotes a certain combination of Virasoro raising operators of total level $rs-1$. These only serve to contribute an overall $(\sigma,k,l)$ independent factor to the one-point

[3] We have rescaled the usual definition of $\mu_B$, together with the rescaling of $\mu$ that we mentioned above (2.11). Our conventions for $\mu$ and $\mu_B$ relative to, e.g. [14], are $\mu_{\rm here} = \pi\mu_{\rm there}\gamma(b^2)$ and $(\mu_B)_{\rm here} = (\mu_B)_{\rm there}\sqrt{\pi\gamma(b^2)\sin\pi b^2}$.

functions. Since $\alpha_{r,s} = \beta_{r,-s}$, the ground ring ring one-point functions are identical to the tachyon one-point functions (3.4) up to normalization.

Finally, let us consider the physical operators at negative ghost number [26].[4] These are essentially copies of the ground ring, and their construction is analogous to (3.5). Their Liouville momentum is also given by $\beta_{r,s}$, but with $s$ taking the values $s < -q$ and $s \neq 0 \bmod q$. Thus their one-point functions will also be given by (3.4) up to normalization, with the appropriate values of $s$.

The tachyons, the ground ring, and the copies of the ground ring at negative ghost number are the complete set of physical operators in the minimal string theory. Their one-point functions (3.4) have several interesting properties as functions of $(\sigma, k, l)$. First, they satisfy an identity relating states with arbitrary matter label to states with matter label $(k, l) = (1, 1)$

$$\langle \mathcal{O} |\, \sigma; k, l \rangle = \sum_{m', n'} \langle \mathcal{O} |\, \sigma + \frac{i(m'q + n'p)}{\sqrt{pq}}; 1, 1 \rangle \tag{3.6}$$

with $(m', n')$ ranging over the values

$$\begin{aligned} m' &= -(k-1), -(k-1)+2, \ldots, k-1 \\ n' &= -(l-1), -(l-1)+2, \ldots, l-1\ . \end{aligned} \tag{3.7}$$

Here $\mathcal{O}$ stands for an arbitrary physical operator. This is evidence that in the full string theory, where the boundary states are representatives of the BRST cohomology, the following is true

$$|\, \sigma; k, l \rangle = \sum_{m', n'} |\, \sigma + \frac{i(m'q + n'p)}{\sqrt{pq}}; 1, 1 \rangle \tag{3.8}$$

modulo BRST exact states. We should emphasize that (3.8), which relates branes with different matter states, is an inherently quantum mechanical result. This relation is difficult to understand semiclassically, where branes with different matter states appear distinct. But there is no contradiction, because (3.8) involves a shift of $\sigma$ by an imaginary quantity, which amounts to analytic continuation of $\mu_B$ from the semiclassical region where it is real and positive.

[4] In some of the literature, physical operators at positive ghost number are also discussed. However, these violate the Liouville bound $\alpha < Q/2$ [42]. Thus they are not distinct from the negative ghost number operators, but are related by the reflection $\alpha \to Q - \alpha$.

According to (3.8), the FZZT branes with $(k,l)=(1,1)$ form a complete basis of all the FZZT branes of the theory. The branes with other matter states should be thought of as multi-brane states formed out of these elementary FZZT branes. This allows us to simplify our discussion henceforth by restricting our attention, without loss of generality, to the elementary FZZT (and ZZ) branes with $(k,l)=(1,1)$. We will also simplify the notation by dropping the label $(1,1)$ from the boundary states; this label will be implicit throughout.

A second interesting property of the one-point functions is that they are clearly invariant under the transformations

$$\sigma \to -\sigma, \quad \sigma \pm 2i\sqrt{pq} \tag{3.9}$$

Again, this is evidence that the states labelled by $\sigma$ should be identified under the transformations (3.9) modulo BRST exact states. Thus, labelling the states by $\sigma \in \mathbb{C}$ infinitely overcounts the number of distinct states. Therefore, it makes more sense to define

$$z = \cosh \frac{\pi\sigma}{\sqrt{pq}} \tag{3.10}$$

and to label the states by $z$,

$$|\,\sigma\rangle \to |\,z\rangle \tag{3.11}$$

such that two states $|\,z\rangle$ and $|\,z'\rangle$ are equal if and only if $z=z'$. In section 4, we will interpret geometrically the parameter $z$ and the infinite overcounting by $\sigma$.

Now let us discuss the ZZ boundary states. As was the case for the FZZT states, these are formed by tensoring together a Liouville ZZ boundary state and a matter Cardy state (in this case the $(1,1)$ matter state). However, here there are subtleties arising from the fact that $b^2$ is rational: the Liouville ZZ states are in one-to-one correspondence with the degenerate representations of Liouville theory, which, as we discussed in section 2, have rather different properties at generic $b$ and at $b^2$ rational. In either case, the prescription [16,18] for constructing the ZZ boundary states is to take the formula for the irreducible character ((2.19) for rational $b^2$) and replace each term $\frac{1}{\eta(q)}q^{-N^2/4pq}$ with an FZZT boundary state with $\sigma = iN$. Thus (2.19) becomes

$$\begin{aligned}|\,t,m,n\rangle &= \sum_{j=0}^{t}\left(\Big|\,z=\cos\frac{\pi N(t-2j,m,n)}{pq}\Big\rangle - \Big|\,z=\cos\frac{\pi N(t-2j,m,-n)}{pq}\Big\rangle\right)\\ &= (t+1)\left(\Big|\,z=(-1)^t\cos\frac{\pi(mq+np)}{pq}\Big\rangle - \Big|\,z=(-1)^t\cos\frac{\pi(mq-np)}{pq}\Big\rangle\right)\end{aligned} \tag{3.12}$$

In the second equation we have substituted (2.14) and simplified the arguments of the cosines – surprisingly, they become independent of $j$. We recognize the quantity in parentheses to be a ZZ state with $t = 0$; thus we conclude that

$$|\, t, m, n\rangle = \begin{cases} +(t+1)|\, t = 0, m, n\rangle & t \text{ even} \\ -(t+1)|\, t = 0, m, q-n\rangle & t \text{ odd} \end{cases} \tag{3.13}$$

It is also straightforward to show using (3.12) that

$$|\, t, m, n\rangle = |\, t, p-m, q-n\rangle \tag{3.14}$$

and that

$$|\, t, m, n\rangle = 0 \quad \text{when} \;\; m = p \;\; \text{or} \;\; n = q \tag{3.15}$$

One should keep in mind that (3.13)–(3.15) are meant to be true modulo BRST null states.

As implied by the comment below (2.20), the states with $t = 0$ appearing in (3.13) are identical to the ZZ boundary states for generic $b$, which can be written as differences of just two FZZT states [18]:

$$\begin{aligned} |\, t = 0, m, n\rangle &= |\, z = \cos\frac{\pi\sigma(m,n)}{\sqrt{pq}}\rangle - |\, z = \cos\frac{\pi\sigma(m,-n)}{\sqrt{pq}}\rangle \\ &= 2\sum_{k',l'}\int_0^\infty dP\, \sinh(\frac{2\pi m P}{b})\sinh(2\pi n P b)\Psi^*(P)\sqrt{S(1,1;k',l')}|P\rangle\rangle_L|k',l'\rangle\rangle_M \end{aligned} \tag{3.16}$$

with

$$\sigma(m,n) = i\left(\frac{m}{b} + nb\right) \; . \tag{3.17}$$

These expressions will be useful below. The boundary cosmological constant corresponding to $\sigma(m,n)$ is

$$\mu_B(m,n) = \sqrt{\mu}\,(-1)^m \cos\pi n b^2 \; . \tag{3.18}$$

Thus the two subtracted FZZT states in (3.16) have the same boundary cosmological constant. In the next section, we will interpret geometrically this fact, together with the formula (3.13) for the general ZZ boundary state.

Using the identifications (3.13)–(3.15), we can reduce any ZZ brane down to a linear combination of $(t = 0, m, n)$ branes with $1 \le m \le p-1$, $1 \le n \le q-1$ and $mq - np > 0$. We will call these $(p-1)(q-1)/2$ branes the *principal ZZ branes.* It is easy to see from (3.16) that the one-point functions of physical operators are sufficient to distinguish the

principal ZZ branes from one another. Thus the principal ZZ branes form a complete and linearly independent basis of physical states with the ZZ-type boundary conditions.

We will conclude this section by discussing an interesting feature of the principal ZZ branes. The ground ring one-point functions in the principal ZZ brane states can be normalized so that (we drop the label $t = 0$ from these states from this point onwards)

$$\langle \widehat{\mathcal{O}}_{r,s} | m, n \rangle = U_{s-1}\left( (-1)^m \cos \frac{\pi n p}{q} \right) U_{r-1}\left( (-1)^n \cos \frac{\pi m q}{p} \right) \langle 0 | m, n \rangle \ , \tag{3.19}$$

where $\langle 0|m,n\rangle$ denotes the ZZ partition function (i.e. the one-point function of the identity operator). This is consistent with the ring multiplication rule

$$\widehat{\mathcal{O}}_{r,s} = U_{s-1}(\widehat{x}) U_{r-1}(\widehat{y}) \tag{3.20}$$

assuming that the principal ZZ branes are eigenstates of the ring generators:

$$\begin{aligned} \widehat{x}|m,n\rangle &= x_{mn}|m,n\rangle \\ \widehat{y}|m,n\rangle &= y_{mn}|m,n\rangle \end{aligned} \tag{3.21}$$

with eigenvalues

$$x_{mn} = (-1)^m \cos \frac{\pi n p}{q}, \qquad y_{mn} = (-1)^n \cos \frac{\pi m q}{p} \ . \tag{3.22}$$

Assuming the principal ZZ branes are eigenstates of the ring elements, the expression (3.19) constitutes an independent derivation of our ansatz (2.30) for the $\mu$-deformed ring multiplication. This derivation allows us to avoid the explicit computation of minimal model and Liouville OPEs that would have otherwise been necessary to obtain (2.30).

Let us make a few more comments on the result (3.21).

1. It is clear that the general FZZT boundary state labelled by $\sigma$ will not be an eigenstate of the ring generators: this property is special to the ZZ boundary states.

2. Once we have normalized the ring elements to bring their one-point functions to the form (3.19), the ZZ branes with other matter labels $(k,l)$ will not be eigenstates of the ring elements. Of course, we could have normalized the ring elements with respect to a different $(k,l)$; then (3.19) would have applied to this matter label. But in view of the decomposition (3.8), it is natural to assume that the branes with matter label $(1,1)$ are eigenstates of the ring.

3. Finally, notice that $x_{mn}$ is essentially the value of the boundary cosmological constant (3.18) associated with the $(m,n)$ ZZ brane. In the next section, we will see what $y_{mn}$ corresponds to.

## 4. Geometric interpretation of minimal string theory

### *4.1. The surface $\mathcal{M}_{p,q}$ and its analytic structure*

In this section, we will provide a geometric interpretation of minimal string theory. We will see how the structure of an auxiliary Riemann surface can explain many of the features of the FZZT and ZZ branes that we found above. Of primary importance will be the partition function $Z$ of the FZZT boundary state. This partition function depends on the bulk and boundary cosmological constants, $\mu$ and $\mu_B$. Differentiating with respect to $\mu$ gives the expectation value of the bulk cosmological constant operator with FZZT boundary conditions:

$$\partial_\mu Z\big|_{\mu_B} = \langle c\bar{c} V_b \rangle\big|_{\text{FZZT}} \ . \tag{4.1}$$

Using the formulas (3.1) and (3.2) for the FZZT boundary state with $P = i(Q/2 - b)$, we obtain

$$\partial_\mu Z\big|_{\mu_B} = \frac{1}{2(b^2-1)} (\sqrt{\mu})^{1/b^2 - 1} \cosh\left(b - \frac{1}{b}\right) \pi\sigma \ , \tag{4.2}$$

where we have fixed the normalization of $Z$ for later convenience. We have also suppressed the dependence on the matter state, since this will be $\mu_B$ and $\mu$ independent. Integrating (4.2) with respect to $\mu$ then gives

$$Z = \frac{b^2}{b^4-1} (\sqrt{\mu})^{1/b^2+1} \left(b^2 \cosh \pi b\sigma \, \cosh\frac{\pi\sigma}{b} - \sinh \pi b\sigma \, \sinh\frac{\pi\sigma}{b}\right) \ . \tag{4.3}$$

Finally, we can differentiate with respect to $\mu_B$ at fixed $\mu$ to obtain the relatively simple formula:

$$\partial_{\mu_B} Z\big|_{\mu} = (\sqrt{\mu})^{1/b^2} \cosh\frac{\pi\sigma}{b} \ . \tag{4.4}$$

The normalization of $Z$ was chosen so that the coefficient of this expression would be unity.

We can similarly consider the dual FZZT brane, which is given in terms of the dual bulk and boundary cosmological constants $\tilde{\mu}$ and $\tilde{\mu}_B$. The former was defined in (2.11), while the latter is given by

$$\frac{\tilde{\mu}_B}{\sqrt{\tilde{\mu}}} = \cosh\frac{\pi\sigma}{b} \ , \tag{4.5}$$

where again we have rescaled the usual definition of $\tilde{\mu}_B$ by a convenient factor. It was observed in [13-16] that the Liouville observables are invariant under $b \to 1/b$ provided one

takes $\mu, \mu_B \to \tilde{\mu}, \tilde{\mu}_B$ as well. Thus the dual brane provides a physically equivalent description of the FZZT boundary conditions. Mapping $b \to 1/b$ and applying the transformation (4.5) to the FZZT partition function (4.3), we find the dual partition function

$$\tilde{Z} = \frac{b^2}{1-b^4}(\sqrt{\tilde{\mu}})^{b^2+1}\left(\frac{1}{b^2}\cosh\pi b\sigma\,\cosh\frac{\pi\sigma}{b} - \sinh\pi b\sigma\,\sinh\frac{\pi\sigma}{b}\right) . \tag{4.6}$$

Note that $\tilde{Z} \neq Z$, although the formula (3.2) for the one-point functions is self-dual under $b \to 1/b$. The two loops are physically equivalent by construction: physical observables can be calculated equivalently with either the FZZT brane or its dual. Differentiating (4.6) with respect to $\tilde{\mu}_B$ (while holding $\tilde{\mu}$ fixed) leads to

$$\partial_{\tilde{\mu}_B}\tilde{Z}\big|_{\tilde{\mu}} = (\sqrt{\tilde{\mu}})^{b^2}\cosh\pi b\sigma \ . \tag{4.7}$$

So far, the discussion has been for general $b$. Now let us consider what happens at the special, rational values of $b^2 = p/q$ that describe the $(p,q)$ minimal string theories. Let us define the dimensionless variables

$$\begin{aligned} x &= \frac{\mu_B}{\sqrt{\mu}} \ , \qquad & y &= \frac{\partial_{\mu_B}Z}{\sqrt{\tilde{\mu}}} \ , \\ \tilde{x} &= \frac{\tilde{\mu}_B}{\sqrt{\tilde{\mu}}} \ , \qquad & \tilde{y} &= \frac{\partial_{\tilde{\mu}_B}\tilde{Z}}{\sqrt{\mu}} \ . \end{aligned} \tag{4.8}$$

Then we can rewrite the equations (4.4) and (4.7) as polynomial equations in these quantities:

$$\begin{aligned} F(x,y) &= T_q(x) - T_p(y) = 0 \\ \tilde{F}(\tilde{x},\tilde{y}) &= T_p(\tilde{x}) - T_q(\tilde{y}) = 0 \ . \end{aligned} \tag{4.9}$$

Therefore, $x$ and $y$ have a natural analytic continuation to a Riemann surface $\mathcal{M}_{p,q}$ described by the curve $F(x,y) = 0$. Moreover, since $\tilde{F}(x,y) = F(y,x)$, the dual FZZT brane gives rise to the *same* Riemann surface. In fact, it is clear from (4.4) and (4.7) that $\tilde{x} = y$ and $\tilde{y} = x$.

The existence of the auxiliary Riemann surface $\mathcal{M}_{p,q}$ suggests that we recast our discussion in a more geometric language. Consider first the FZZT branes. The definition (4.8) of $y$ and $\tilde{y}$ implies that we can think of the FZZT partition function and its dual as integrals of the one-forms $y\,dx$ and $x\,dy$:

$$Z(\mu_B) = \mu^{\frac{p+q}{2p}}\int_{\mathcal{P}}^{x(\mu_B)} y\,dx \ , \qquad \tilde{Z}(\tilde{\mu}_B) = \tilde{\mu}^{\frac{p+q}{2q}}\int_{\mathcal{P}}^{y(\tilde{\mu}_B)} x\,dy \tag{4.10}$$

for some arbitrary fixed point $\mathcal{P} \in \mathcal{M}_{p,q}$. The fact that $Z \neq \tilde{Z}$ is simply the statement that $y\,dx$ and $x\,dy$ are distinct one-forms on $\mathcal{M}_{p,q}$. It is true, however, that they are related by an exact form:

$$y\,dx + x\,dy = d(xy) \ . \tag{4.11}$$

It is not surprising then that the brane and the dual brane are physically equivalent descriptions of the FZZT boundary conditions.

Integrating (4.11) leads to another interesting result. We can think of $\int y\,dx$ and $\int x\,dy$ as effective potentials $V_{eff}(x)$ and $\tilde{V}_{eff}(y)$. Then (4.11) implies that these effective potentials are related by a Legendre transform:

$$V_{eff}(x) = xy - \tilde{V}_{eff}(y) \ . \tag{4.12}$$

It seems then that $x$ and $y$ play the role of coordinate and conjugate momentum on our Riemann surface. We will comment on this interpretation some more when we discuss the matrix model description in section 9.

We can also give a geometric interpretation to the parameter $z$ defined in (3.10). Notice that $z$ is related to the coordinates of our surface via

$$x = T_p(z), \quad y = T_q(z) \tag{4.13}$$

Thus $z$ can be thought of as a uniformizing parameter of $\mathcal{M}_{p,q}$ that covers it exactly once. The parameter $\sigma$ also gives a uniformization of the surface, but it covers the surface infinitely many times. This is consistent with what we saw in section 3, that the parametrization of the FZZT branes by $\sigma$ is redundant, while the $z$ parametrization is unambiguous. We conclude that points on the surface $\mathcal{M}_{p,q}$ are in one to one correspondence with the distinct branes.

In terms of the parameter $z$, the surface $\mathcal{M}_{p,q}$ appears to have genus zero, since $z$ takes values on the whole complex plane. However, there are special distinct values of $z$ that correspond to the same point $(x, y)$ in $\mathcal{M}_{p,q}$. Such points are singularities of $\mathcal{M}_{p,q}$, and they can be thought of as the pinched $A$-cycles of a higher genus surface. It is easy to see that the singularities correspond to the following points in $\mathcal{M}_{p,q}$:

$$(x_{mn}, y_{mn}) = \left( (-1)^m \cos\frac{\pi n p}{q} \ , \ (-1)^n \cos\frac{\pi m q}{p} \right) , \tag{4.14}$$

which come from pairs $z^+_{mn}$ and $z^-_{mn}$, with

$$z^{\pm}_{mn} = \cos \frac{\pi(mq \pm np)}{pq} \tag{4.15}$$

with $m$ and $n$ ranging over

$$m = 1, \ldots, p-1, \quad n = 1, \ldots, q-1, \quad mq - np > 0 \tag{4.16}$$

Therefore there are exactly $(p-1)(q-1)/2$ singularities of $\mathcal{M}_{p,q}$.

These singularities can also be found directly from the curve (4.9) by solving the equations

$$F(x,y) = 0 \ , \quad \partial_x F(x,y) = \partial_y F(x,y) = 0 \ . \tag{4.17}$$

It is straightforward to check that this leads to the same conclusion as (4.14).

With this understanding of the singularities of $\mathcal{M}_{p,q}$, the geometric interpretation of the ZZ branes is clear. Recall that the ZZ branes were found to be eigenstates of the ground ring generators, with eigenvalues given in (3.21). It is easy to see that these eigenvalues lie on the curve $F(x,y) = 0$; therefore the ground ring generators measure the location of the principal ZZ branes on $\mathcal{M}_{p,q}$. Moreover, comparing (3.21) with the singularities (4.14), we find that they are the same. Therefore, the principal ZZ branes are located at the singularities of $\mathcal{M}_{p,q}$!

In fact, since each principal brane is located at a different singularity, and there are exactly as many such branes as there are singularities, every singularity corresponds to a unique principal ZZ brane and vice versa. All of the other $(t,m,n)$ ZZ branes are related to multiples of principal branes by a BRST exact state, and thus the location of a ZZ brane on $\mathcal{M}_{p,q}$ determines its BRST cohomology class. Notice that branes that do not correspond to singularities, namely those with $m = 0 \bmod p$ or $n = 0 \bmod q$, are themselves BRST null.

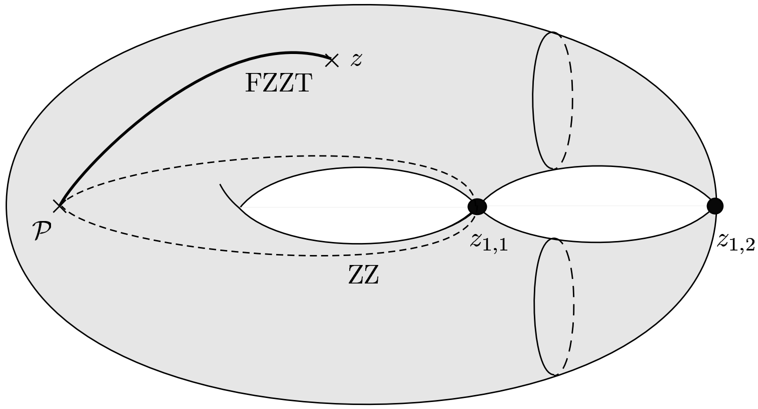


**Fig. 1:** The surface $\mathcal{M}_{p,q}$, along with examples of FZZT and ZZ brane contours, shown here for $(p,q) = (2,5)$. Here $\mathcal{P}$ is an arbitrary, fixed point in $\mathcal{M}_{p,q}$, while $z$ is the uniformizing parameter of $\mathcal{M}_{p,q}$ that corresponds through (4.13) to the boundary cosmological constant and the dual boundary cosmological constant of the associated FZZT brane. The points $z_{1,1}$ and $z_{1,2}$ are the pinched $A$-cycles of the surface, and they correspond to the $(1,1)$ and $(1,2)$ principal ZZ branes. The $(m,n)$ ZZ contour is clearly a difference of two FZZT contours beginning at $\mathcal{P}$ and ending at $z_{mn}$.

The relation (3.16) between the FZZT and the principal ZZ branes also has a simple geometric interpretation. The fact that the FZZT branes are equivalent to line integrals of $y\,dx$ on $\mathcal{M}_{p,q}$ implies that the ZZ branes can be thought of as *periods* of $y\,dx$ around cycles $B_{m,n}$ of $\mathcal{M}_{p,q}$ that run between the singularity $(x_{mn}, y_{mn})$ and an open cycle of $\mathcal{M}_{p,q}$:

$$Z_{m,n} = \mu^{\frac{p+q}{2p}} \oint_{B_{m,n}} y\,dx \ . \tag{4.18}$$

This generalizes an observation in [24]. An illustration of the surface $\mathcal{M}_{p,q}$ and the various contour integrals is shown in figure 1. Note that since the ZZ branes are defined using closed contours, they will be the same whether we use the FZZT brane or its dual. Thus, the (physically equivalent) dual brane does not lead to an overcounting of the ZZ brane spectrum. Actually, the relation between FZZT and principal ZZ boundary states implies a stronger statement than (4.18):

$$|m,n\rangle_{\rm ZZ} = \oint_{B_{m,n}} \partial_x |z(x)\rangle_{\rm FZZT}\ dx \ . \tag{4.19}$$

That is, we can still think of the ZZ branes as periods on $\mathcal{M}_{p,q}$ of the FZZT branes, even at the more general level of boundary states.

Our geometrical interpretation implies that all of the distinct principal ZZ branes exist and play a role in minimal string theory. We can also ask for the interpretation of the general $(t, m, n)$ ZZ brane (let us keep the matter state $(1, 1)$ for simplicity). From (3.13), the answer is clear. The $(t, m, n)$ ZZ brane is simply $(t + 1)$ copies of a closely related principal ZZ brane, and therefore it corresponds to a contour integral of $y\,dx$ that winds $(t + 1)$ times around the $B$-cycle of the principal ZZ brane.

We should point out that a proposal for the interpretation of the $(1, n)$ branes in the $c = 1$ matrix model was recently advanced [53]. It would be interesting to understand its relation to our work.

### *4.2. Deformations of $\mathcal{M}_{p,q}$*

So far, the discussion has been entirely in the "conformal background" where $\mu \geq 0$ and we have a simple world-sheet description of the theory. We have seen that in this case, the surface $\mathcal{M}_{p,q}$ is given by a very simple curve $T_q(x) - T_p(y) = 0$. Here we will consider deformations of this curve, and for simplicity we will require that they simply shift the $N_{p,q} = (p - 1)(q - 1)/2$ singularities of $\mathcal{M}_{p,q}$:

$$(x_i, y_i) \to (x_i + \delta x_i, y_i + \delta y_i) \ , \tag{4.20}$$

without changing their total number. This leads to constraints on the form of the deformations. To see what these constraints are, it is sufficient to work to linear order in the deformation. Before the deformation, the singularities satisfy the following equations:

$$\begin{aligned} &T_q(x_i) - T_p(y_i) = 0 \\ &T_q'(x_i) = T_p'(y_i) = 0 \ . \end{aligned} \tag{4.21}$$

After the deformation, the first equation becomes

$$\begin{aligned} 0 &= T_q(x_i + \delta x_i) - T_p(y_i + \delta y_i) + \delta F(x_i, y_i) \\ &= T_q(x_i) - T_p(y_i) + T_q'(x_i)\delta x_i - T_p'(y_i)\delta y_i + \delta F(x_i, y_i) \ . \end{aligned} \tag{4.22}$$

Using (4.21), we see that the deformation must vanish at the singularities. This gives $N_{p,q}$ constraints on the polynomial $\delta F(x, y)$. The deformation of the second and third equations of (4.21) gives no further constraints; instead, solving them yields formulas for $\delta x_i$ and $\delta y_i$.

Intuitively, we expect such deformations of $\mathcal{M}_{p,q}$ to correspond in the minimal string theory to perturbations of the background by physical vertex operators:

$$\begin{aligned} &\delta S = t_{r,s} V_{r,s} \\ &r = 1, \ldots, p-1, \quad s \leq q-1, \quad s \neq 0 \bmod q, \quad qr - ps > 0 \ . \end{aligned} \tag{4.23}$$

These vertex operators exist at all ghost numbers $\leq 1$ [26]. They have KPZ scaling

$$t_{r,s} \sim \mu^{\frac{p+q-qr+ps}{2p}} \tag{4.24}$$

and their Liouville momenta are given by (2.13), with the range of $s$ extended as in (4.23).

In order to relate these perturbations to the deformations of $\mathcal{M}_{p,q}$, notice that to leading order in $t_{r,s}$, the change in the FZZT partition function under the perturbation (4.23) is simply the one-point function $t_{r,s}\langle V_{r,s}\rangle$ in the FZZT boundary state. Using (3.1), we find that

$$\delta Z = t_{r,s}\langle V_{r,s}\rangle = t_{r,s}\mu^{\frac{qr-ps}{2p}} \cosh\left(\frac{qr-ps}{p}\right)\pi b\sigma \ . \tag{4.25}$$

(The matter wavefunction will be irrelevant for this calculation, as will be other overall normalization factors.) Since $y = \mu^{-\frac{p+q}{2p}}\partial_x Z$, we find that if we hold $x$ fixed, then $y$ is deformed by

$$\delta y \sim \tilde{t}_{r,s} \frac{\sinh\left(\frac{qr-ps}{p}\right)\pi b\sigma}{\sinh \pi b\sigma} \ , \tag{4.26}$$

where we have defined the dimensionless parameter $\tilde{t}_{r,s} = t_{r,s}\mu^{\frac{qr-ps-p-q}{2p}}$. In terms of $x$ and $y$, the deformation of the curve is then

$$\delta_{r,s}F(x,y) = p\, U_{p-1}(y)\delta y \sim \tilde{t}_{r,s}\Big(U_{q-1}(x)T_s(x)U_{r-1}(y) - U_{p-1}(y)T_r(y)U_{s-1}(x)\Big) \ . \tag{4.27}$$

Here we have extended the definition of the Chebyshev polynomials to negative $s$ in a natural way: $U_{-s-1}(x) = -U_{s-1}(x)$. In particular, $U_{-1}(x) = 0$. Finally, using the original curve $F(x,y) = 0$, we can obtain a more compact formula for the deformations of $\mathcal{M}_{p,q}$ corresponding to the bulk physical operators (4.23):

$$\begin{aligned} &\delta_{r,s}F(x,y) = \tilde{t}_{r,s}\Big(U_{q-s-1}(x)U_{r-1}(y) - U_{p-r-1}(y)U_{s-1}(x)\Big) \\ &r = 1, \ldots, p-1 \ , \quad s \leq q-1 \ , \quad s \neq 0 \bmod q \ , \quad qr - ps > 0 \ . \end{aligned} \tag{4.28}$$

These deformations of $\mathcal{M}_{p,q}$ indeed vanish on the singularities, where $U_{q-1}(x) = U_{p-1}(u) = 0$. Therefore, all of the perturbations of the string theory background by bulk physical vertex operators correspond to singularity-preserving deformations of $\mathcal{M}_{p,q}$.

It is interesting to ask whether the converse is also true, i.e. whether every possible singularity-preserving deformation of $\mathcal{M}_{p,q}$ is given by (4.28). A careful analysis reveals that a complete basis of such deformations is indeed given by the same formula (4.28) for $\delta_{r,s}F$, but with $r$ and $s$ taking values in a slightly larger range:

$$r = 1, \ldots, p \ , \quad s \leq q-1 \ , \quad qr - ps > 0 \ . \tag{4.29}$$

Thus, there are deformations of $\mathcal{M}_{p,q}$ that do not correspond to perturbations by bulk physical vertex operators. These have $r = p$ or $s = 0 \bmod q$. It is easy to see that the former category can be thought of as reparametrizations of $y$ alone:

$$y \to y + \tilde{t}_{p,s} U_{q-s-1}(x) \ , \quad s \leq q-1 \ . \tag{4.30}$$

Deformations with $s = 0 \bmod q$ are more complicated, but they also correspond to polynomial reparametrizations of $x$ alone.

The worldsheet interpretation of these extra deformations with $r = p$ or $s = 0 \bmod q$ is not always clear. Recall however that the surface $\mathcal{M}_{p,q}$ originally arose from the disk amplitude of the FZZT brane. It is reasonable then to expect the deformations associated with such reparametrizations to be associated with open strings on the FZZT brane. For example, $U_{q-1}(x)$ is the boundary length operator as in [54]. A more detailed correspondence would require a better understanding of open minimal string theory. While this is certainly an interesting problem, it is beyond the scope of this paper.

In order to examine the relevance of the perturbations we assign weight $(p, q)$ to $(x, y)$ so that $F(x, y)$ is a quasi-homogeneous polynomial of degree $pq$, the degree of $\delta_{r,s}F(x, y)$ is always $q(r-1) + p(q-s-1)$. This is less than the degree of $F(x, y)$ only for

$$rq - sp < p + q \ . \tag{4.31}$$

Such deformations of $\mathcal{M}_{p,q}$ become important at small $x$ and $y$. In the string theory, they correspond to perturbations by tachyons $\mathcal{T}_{r,s}$ with positive KPZ scaling. These are, of course, precisely the perturbations that become increasingly relevant in the IR. This agrees well with the intuition that flowing to the IR in the string theory corresponds to taking $x$ and $y \to 0$ in $\mathcal{M}_{p,q}$.

So far we have limited ourselves to deformations which do not open the singularities to smooth $A$ cycles. It is then natural to ask what deformations which smooth the singularities correspond to. Given that the $(m, n)$ ZZ brane is associated with the $(m, n)$ singularity, it is reasonable to expect that a background with $(m, n)$ ZZ branes is described by a Riemann surface with the cycle $A_{m,n}$ smoothed out. Since a small number of D-branes does not affect classical string theory, we expect that the number of D-branes $N_{m,n}$ needed to change the Riemann surface to be of order $1/g_s$. This number can be measured by computing the period of the one form $ydx$ around the $A$ cycle

$$\oint_{A_{m,n}} ydx = g_s N_{m,n} \ . \tag{4.32}$$

The cycle $A_{m,n}$ is conjugate to $B_{m,n}$ because they have intersection number one. Therefore, the ZZ brane creation operator (4.19) which is an integral around $B_{m,n}$ changes $N_{m,n}$ which is conjugate to it by one unit. This interpretation of smoothing the $(m, n)$ singularity is similar to the picture which has emerged in recent studies of four dimensional gauge theories and matrix models [55,56].

## 5. Minimal superstring theory

### *5.1. Preliminaries*

We now turn to the study of minimal superstring theory. We start by reviewing the $(p, q)$ superminimal models. For these theories, it will be more convenient to work in $\alpha' = 2$ units. As in the bosonic theories, one must have $p, q \geq 2$. Moreover, one must have either $(p, q)$ odd and coprime, or else $(p, q)$ even, $(p/2, q/2)$ coprime, and $(p-q)/2$ odd (the last condition follows from modular invariance [57,24]). The superminimal models have central charge

$$\widehat{c} = 1 - \frac{2(p-q)^2}{p\,q} \tag{5.1}$$

As in the bosonic theories, primary operators $\mathcal{O}_{r,s}$ are labelled by integers $r$ and $s$ with $r = 1, \ldots, p-1$ and $s = 1, \ldots, q-1$ and $\mathcal{O}_{p-r,q-s} \equiv \mathcal{O}_{r,s}$. The crucial difference here however is that we must distinguish between NS and the R sector operators. The NS (R) operators have $r - s$ even (odd). As a result, the operator dimensions are given by a slightly more complicated expression:

$$\Delta(\mathcal{O}_{r,s}) = \overline{\Delta}(\mathcal{O}_{r,s}) = \frac{(rq-sp)^2 - (p-q)^2}{8pq} + \frac{1-(-1)^{r-s}}{32} \ . \tag{5.2}$$

Of particular interest is the operator $\mathcal{O}_{\frac{p}{2},\frac{q}{2}}$ which exists only in the $(p,q)$ even theories. It corresponds to the supersymmetric Ramond ground state. Since it is absent in the $(p,q)$ odd theories, they break supersymmetry.

The operators of the $(p,q)$ superminimal models obey fusion rules identical to those of the bosonic models (2.3). Note that the "generators" $\mathcal{O}_{1,2}$ and $\mathcal{O}_{2,1}$ of the $(p,q)$ superminimal models are in the R sector, and thus the fusion of these operators with a general R (NS) primary results in a set of NS (R) operators.

Now consider super-Liouville theory with central charge

$$\widehat{c} = 1 + 2Q^2 = 1 + 2\left(b + \frac{1}{b}\right)^2 . \tag{5.3}$$

The basic vertex operators of super-Liouville are the NS operators $N_\alpha = e^{\alpha\phi}$ and the R operators $R^\pm_\alpha = \sigma^\pm e^{\alpha\phi}$. These have dimensions

$$\begin{aligned} \Delta(N_\alpha) = \overline{\Delta}(N_\alpha) &= \frac{1}{2}\alpha(Q-\alpha) \\ \Delta(R^\pm_\alpha) = \overline{\Delta}(R^\pm_\alpha) &= \frac{1}{2}\alpha(Q-\alpha) + \frac{1}{16} . \end{aligned} \tag{5.4}$$

Here $\sigma^\pm$ denotes the dimension 1/16 spin fields of the super-Liouville theory. If we study super-Liouville as an isolated quantum field theory, only one of these fields exists following the GSO projection, say $R^-_\alpha$. However, if we combine it with another "matter" theory, we sometimes need both of them before performing a GSO projection on the combined theory.

The supersymmetric Ramond ground state has $\alpha = \frac{Q}{2}$ and hence $\Delta = \frac{\widehat{c}}{16}$. It does not correspond to two degenerate fields $R^\pm$ which are related by the action of the supercharge. Instead, from solving the minisuperspace equations we find two possible wave functions $\psi^+ = e^{\mu e^{b\phi}}$ with $\psi^- = 0$, or $\psi^- = e^{-\mu e^{b\phi}}$ with $\psi^+ = 0$ [23]. Imposing that the wave function goes to zero as $\phi \to +\infty$, we see that depending on the sign of $\mu$

$$\zeta = \mathrm{sign}(\mu) , \tag{5.5}$$

we have only one wave function $\psi^{-\zeta} = e^{-|\mu| e^{b\phi}}$ and $\psi^\zeta = 0$. Therefore, in pure super-Liouville theory, where we keep only $R^-_{\alpha=\frac{Q}{2}}$, the Ramond ground state exists only for positive $\mu$. Below we will review how this conclusion changes when we add the matter theory.

The degenerate primaries of super-Liouville are also similar to those of ordinary Liouville. These are given by

$$\begin{aligned} N_{\alpha_{r,s}} &= e^{\alpha_{r,s}\phi} \ , \quad r-s \in 2\mathbb{Z} \\ R^{\pm}_{\alpha_{r,s}} &= \sigma^{\pm} e^{\alpha_{r,s}\phi} \ , \quad r-s \in 2\mathbb{Z}+1 \\ 2\alpha_{r,s} &= \frac{1}{b}(1-r) + b(1-s) \ . \end{aligned} \tag{5.6}$$

The analogue of the bosonic fusion rules (2.9), (2.10) are for super-Liouville theory [58-61]:

$$\begin{aligned} R_{-\frac{b}{2}} R_{\alpha} &= [N_{\alpha-\frac{b}{2}}] + C_{-}^{(R)}(\alpha)[N_{\alpha+\frac{b}{2}}] \\ R_{-\frac{b}{2}} N_{\alpha} &= [R_{\alpha-\frac{b}{2}}] + C_{-}^{(NS)}(\alpha)[R_{\alpha+\frac{b}{2}}] \\ C_{-}^{(R)}(\alpha) &= \frac{1}{4}\mu b^2 \frac{\gamma(\alpha b - \frac{b^2}{2})}{\gamma\left(\frac{1-b^2}{2}\right)\gamma\left(\alpha b + \frac{1}{2}\right)} \\ C_{-}^{(NS)}(\alpha) &= \frac{1}{4}\mu b^2 \frac{\gamma(\alpha b - \frac{b^2}{2} - \frac{1}{2})}{\gamma\left(\frac{1-b^2}{2}\right)\gamma\left(\alpha b\right)} \end{aligned} \tag{5.7}$$

with similar expressions for $R_{-\frac{1}{2b}}$ with $b \to 1/b$ and $\mu \to \tilde{\mu}$, where $\tilde{\mu}$ is the dual cosmological constant $\pi\tilde{\mu}\gamma(\frac{Q}{2b}) = (\pi\mu\gamma(\frac{bQ}{2}))^{1/b^2}$. As in the bosonic string, we will rescale $\mu$ and $\tilde{\mu}$ so that they are more simply related

$$\tilde{\mu} = \mu^{1/b^2} \ . \tag{5.8}$$

We now couple the super-minimal matter theory to the super-Liouville theory to form the minimal superstring. Imposing that the combined system has the correct central charge fixes the Liouville parameter

$$b = \sqrt{\frac{p}{q}} \tag{5.9}$$

The tachyon operators $\mathcal{T}_{r,s}$ are obtained by dressing the matter operators $\mathcal{O}_{r,s}$ with

$$\begin{aligned} N_{\beta_{r,s}} &= e^{\beta_{r,s}\phi} \ , \quad r-s \in 2\mathbb{Z} \\ R^{\pm}_{\beta_{r,s}} &= \sigma^{\pm} e^{\beta_{r,s}\phi} \ , \quad r-s \in 2\mathbb{Z}+1 \\ 2\beta_{r,s} &= \frac{p+q-rq+sp}{\sqrt{p\,q}} \ , \quad rq-sp \geq 0 \end{aligned} \tag{5.10}$$

and the appropriate superghosts.

The matter Ramond operators are labelled by their fermion number $\mathcal{O}^{\pm}_{r,s}$. (In the superminimal model without gravity we keep only one of them, say $\mathcal{O}^{+}_{r,s}$.) This fermion number should be correlated with the fermion number in super-Liouville. However, there

is an important subtlety which should be explained here. So far we discussed two fermion number operators $(-1)^{f^{L,M}}$, one in Liouville and one in the matter sector of the theory. Their action on the lowest energy states in a Ramond representation is proportional to $iG_0^{L,M}\tilde{G}_0^{L,M}$. When we combine the Liouville and the matter states it is natural to define left and right moving fermion numbers $(-1)^{f_L}$ and $(-1)^{f_R}$ such that their action on the lowest energy states in a Ramond representation is proportional to $iG_0^L G_0^M$, and $i\tilde{G}_0^L\tilde{G}_0^M$ respectively. Then it is conventional to define the total fermion number as $(-1)^f = (-1)^{f_L+f_R}$. Therefore,

$$(-1)^f = (-1)^{f_L+f_R} = (-1)^{f^L+f^M+1} \ , \tag{5.11}$$

i.e. the total (left plus right) worldsheet fermion number differs from the sum of the fermion numbers in the Liouville and the matter parts of the theory.

Consider now the vertex operators in the $(-1/2,-1/2)$ ghost picture. In the 0B theory we project on $(-1)^f = 1$. Therefore, following (5.11) we project on $(-1)^{f^L+f^M} = -1$. The candidate operators are $\mathcal{O}^{\pm}_{r,s}R^{\mp}_{\beta_{r,s}}$ (we suppress the ghosts). Only one linear combination of them is physical [24]. The situation is more interesting when we try to dress the supersymmetric matter groundstate $(r,s) = (\frac{p}{2},\frac{q}{2})$. Now there is only one matter operator,[5] say $\mathcal{O}^{+}_{\frac{p}{2},\frac{q}{2}}$. Hence it should be dressed with $R^{-}_{\frac{Q}{2}}$. But as we said above, this operator exists only for positive $\mu$. Therefore, in the 0B minimal superstring theories with $(p,q)$ even, the Ramond ground state exists only for $\mu > 0$. For $(p,q)$ odd, there is of course no Ramond ground state to begin with.

The situation is somewhat different in the $(-1/2,-3/2)$ or $(-3/2,-1/2)$ ghost picture, where because of (5.11), we project on $(-1)^{f^L+f^M} = +1$. For generic $(r,s)$ we can use inverse picture changing to find a single vertex operator which is a linear combination of $\mathcal{O}^{\pm}_{r,s}R^{\pm}_{\beta_{r,s}}$. The orthogonal linear combination is a gauge mode which is annihilated when we try to picture change to the $(-1/2,-1/2)$ picture. There are new subtleties for the Ramond ground state $(r,s) = (\frac{p}{2},\frac{q}{2})$. The operator $\mathcal{O}^{+}_{\frac{p}{2},\frac{q}{2}}R^{+}_{\frac{Q}{2}}$ exists only for negative $\mu$. If we try to picture change it to the $(-1/2,-1/2)$ picture we find zero. But there could be another operator $\mathcal{O}^{+}_{\frac{p}{2},\frac{q}{2}}\tilde{R}^{+}_{\frac{Q}{2}}$, which exists only for positive $\mu$ and is related by picture changing to the Ramond ground state $\mathcal{O}^{+}_{\frac{p}{2},\frac{q}{2}}R^{-}_{\frac{Q}{2}}$ we found in the $(-1/2,-1/2)$ picture. The wave function of $\tilde{R}^{+}_{\frac{Q}{2}}$ is obtained by solving the equation

$$(\partial_\phi - b\mu e^{b\phi})\tilde{\psi}^{+} = e^{-\mu e^{b\phi}} \tag{5.12}$$

[5] This fact is not true in other systems like the $\widehat{c} = 1$ theory.

subject to the boundary condition that $\tilde{\psi}^+(\phi \to +\infty) = 0$. The answer is given by an incomplete Gamma function

$$\tilde{\psi}^+ = -\frac{e^{-\mu e^{b\phi}}}{b}\int_0^\infty \frac{e^{-2t}}{t+\mu e^{b\phi}}dt = \begin{cases} \phi + \dots & \phi \to -\infty \\ -\frac{1}{2b\mu}e^{-b\phi}e^{-\mu e^{b\phi}}(1+\dots) & \phi \to +\infty \ . \end{cases} \tag{5.13}$$

The asymptotic form as $\phi \to -\infty$ leads to the form of the operator $\tilde{R}^+_{\frac{Q}{2}} = \sigma^+\phi e^{\frac{Q}{2}\phi}$. Because of the factor of $\phi$, this is not a standard Liouville operator and its analysis is subtle. It is possible, however, to picture change it to the $(-1/2,-1/2)$ picture, where it is simple.

In the spacetime description of these theories, each operator is the "on-shell mode" of a field which depends on the Liouville coordinate $\phi$. Ramond vertex operators in the $(-1/2,-3/2)$ or $(-3/2,-1/2)$ picture describe the RR scalar $C$, while operators in the $(-1/2,-1/2)$ picture describe its gradient $\partial_\phi C$ (more precisely, $(\partial_\phi - b\mu e^{b\phi})C$ [24]). Thus in the spacetime description, the first candidate Ramond ground state $\mathcal{O}^+_{\frac{p}{2},\frac{q}{2}}R^+_{\frac{Q}{2}}$, which exists only for $\mu < 0$ and is zero in the $(-1/2,-1/2)$ picture, describes the constant mode of $C$. It decouples from all correlation functions of local vertex operators but is important in the coupling to D-branes. On the other hand, the second candidate operator for the Ramond ground state $\mathcal{O}^+_{\frac{p}{2},\frac{q}{2}}\tilde{R}^+_{\frac{Q}{2}}$, which exists only for $\mu > 0$, corresponds (asymptotically as $\phi \to -\infty$) to changes of $\partial_\phi C$, i.e. it describes changes in RR flux. Therefore its wave function is linear in $\phi$ in the $(-1/2,-3/2)$ picture, and it is constant in the $(-1/2,-1/2)$ picture. To summarize, in the bulk we have for $\mu > 0$ a Ramond ground state operator that describes changes in RR flux, but no such operator for $\mu < 0$. But in the presence of D-branes, there exists for $\mu < 0$ a Ramond ground state operator that couples to D-brane charge. In other words, for $\mu > 0$ we can have RR flux but no charged D-branes, while the opposite is true for $\mu < 0$.

All of these 0B theories have a global $\mathbb{Z}_2$ symmetry which acts as $-1$ $(+1)$ on all R (NS) operators. We refer to this operation as $(-1)^{F_L}$ where $F_L$ is the left-moving spacetime fermion number. Orbifolding by this symmetry yields the 0A theories. In the 0A theories, all of the physical operators in the Ramond sector are projected out. The only physical operators from the twisted sector are constructed out of the Ramond ground state ($r = \frac{p}{2}, s = \frac{q}{2}$). They exist only when such an operator could not be dressed in the 0B theory. Thus in 0A, we can have RR flux but no charged D-branes for $\mu < 0$ and the opposite for $\mu > 0$.

Finally we should discuss the operation of $(-1)^{f_L}$ with $f_L$ the left-moving worldsheet fermion number. This operation is an $R$-transformation because it does not commute with the supercharge. However, it is not a symmetry of the theory for two reasons. First, the cosmological constant term in the worldsheet action explicitly breaks the would-be symmetry: acting with $(-1)^{f_L}$ sends $\mu \to -\mu$. The second reason this symmetry is broken is that the transformation by $(-1)^{f_L}$ reverses the way the GSO projection is applied in the Ramond sector. We saw that in the 0B theory, one linear combination of the matter operator $\mathcal{O}^{\pm}_{r,s} R^{\mp}_{\beta_{r,s}}$ was physical. After the transformation by $(-1)^{f_L}$, the orthogonal linear combination will be physical. As usual, the situation is more complicated for the Ramond ground state. The $(p,q)$ odd models do not have a Ramond ground state; therefore the spectrum of physical states before and after this operation are identical. Hence, we expect that in these models, the theory with $\mu$ is dual to the theory with $-\mu$. Moreover, at $\mu = 0$ the operation of $(-1)^{f_L}$ becomes a symmetry of the theory.[6] On the other hand, the $(p,q)$ even models have a Ramond ground state, so the spectrum at $\mu$ differs from the spectrum at $-\mu$. An interesting special case is the pure supergravity theory ($p = 2, q = 4$) where the 0B theory at $\mu$ is the same as the 0A theory at $-\mu$ [24]. For the $(p,q)$ even models, the transformation $(-1)^{f_L}$ is generally not useful.

### *5.2. The ground ring of minimal superstring theory*

As in the bosonic string, the minimal superstring theories have a ground ring consisting of all dimension 0, ghost number 0 operators in the BRST cohomology of the theory. Since this ground ring is nearly identical to that of the bosonic string, our discussion will be brief. We will start with the 0B models. Just as in the bosonic string, the ground ring has $(p-1)(q-1)$ elements. The operator $\widehat{\mathcal{O}}_{r,s}$ has Liouville momentum $\alpha_{r,s}$ given by (5.6), and it is constructed by acting on the product $\mathcal{O}_{r,s} V_{\alpha_{r,s}}$ with some combination of raising operators [62-65]. Operators with $r-s$ even (odd) are in the NS (R) sectors. We point out that since we use the Liouville momenta $\alpha_{r,s}$ of (5.6) rather than $\beta_{r,s}$ of (5.10), there is no subtlety associated with the dressing of the Ramond ground state.

When $\mu = 0$, Liouville momentum is conserved in the OPE, and thus one expects on kinematical grounds that just as in the bosonic models, the ground ring is generated by the R sector operators $\mathcal{O}_{1,2}$ and $\mathcal{O}_{2,1}$:

$$\widehat{\mathcal{O}}_{r,s} = \widehat{\mathcal{O}}_{1,2}^{s-1} \widehat{\mathcal{O}}_{2,1}^{r-1} \tag{5.14}$$

[6] A similar situation exists in the $\widehat{c} = 1$ theory where the Ramond ground state appears twice with two different fermion numbers [23].

with the ring relations

$$\widehat{\mathcal{O}}_{1,2}^{q-1} = \widehat{\mathcal{O}}_{2,1}^{p-1} = 0 \ . \tag{5.15}$$

For $\mu \neq 0$, Liouville momentum is no longer conserved, and instead one has super-Liouville fusion rules (5.7) very similar to those of the bosonic theory. We expect the expressions for the ring elements (5.14) and the relations (5.15) to be modified in the same way as in the bosonic string. Once again it will be convenient to define dimensionless generators $\widehat{x}$ and $\widehat{y}$ as in (2.24). Then (5.14) becomes

$$\widehat{\mathcal{O}}_{r,s} = \mu^{\frac{q(r-1)+p(s-1)}{2p}} U_{s-1}(\widehat{x}) U_{r-1}(\widehat{y}) \ , \tag{5.16}$$

while the relations (5.15) become

$$U_{q-1}(\widehat{x}) = U_{p-1}(\widehat{y}) = 0 \ . \tag{5.17}$$

Let us now discuss the relations in the tachyon module. As in the bosonic theory we have

$$\mathcal{T}_{r,s} = \mu^{1-s} \widehat{\mathcal{O}}_{r,s} \mathcal{T}_{1,1} = \mu^{\frac{q(r-1)-p(s-1)}{2p}} U_{s-1}\big(\widehat{x}\big) U_{r-1}\big(\widehat{y}\big) \mathcal{T}_{1,1} \ . \tag{5.18}$$

The Ramond ground state in the even $(p,q)$ models leads to a new complication. As we have seen, the tachyon $\mathcal{T}_{\frac{p}{2},\frac{q}{2}}$ exists in the 0B theory only for positive $\mu$. (For the discussion of correlation functions of physical vertex operators without D-branes we can neglect the similar operator in the $(-1/2,-3/2)$ ghost picture which exists only for negative $\mu$.) Therefore, for positive $\mu$ we have tachyons with $rq - sp \geq 0$, while for negative $\mu$ we have only the tachyons with $rq - sp > 0$. This truncation can be achieved by imposing

$$\mathcal{T}_{p-r,q-s} = \zeta \mu^{\frac{ps-qr}{p}} \mathcal{T}_{r,s} \ . \tag{5.19}$$

As in the bosonic string, it is enough to require that

$$\left(U_{q-2}(\widehat{x}) - \zeta U_{p-2}(\widehat{y})\right) \mathcal{T}_{1,1} = 0 \ . \tag{5.20}$$

This will guarantee all of the relations (5.19).

The discussion of the ground ring and the tachyon module has so far been entirely for the 0B models. The ground ring for 0A follows trivially from that of the 0B: since we need not worry about the Ramond ground state, we simply project out the R sector of the 0B ground ring to obtain the ground ring of 0A. Thus the ground ring of 0A is generated by the NS operators $\widehat{\mathcal{O}}_{1,3}$, $\widehat{\mathcal{O}}_{2,2}$ and $\widehat{\mathcal{O}}_{3,1}$.

It is straightforward to extend the calculation of tachyon correlation functions in the bosonic string (section 2.2) to the superstring. Since the results are nearly identical to the bosonic string, we will not discuss them here.

## 6. FZZT and ZZ branes of minimal superstring theory

### *6.1. Boundary states*

Here we will extend the discussion of the FZZT and the ZZ branes of the previous sections to the $(p,q)$ superminimal models coupled to gravity. The analysis will become more complicated, owing to the presence of the NS and R sectors, the two different GSO projections, and the option of having negative $\mu$. Nevertheless, we will find that the supersymmetric and bosonic theories share many features in common.

As in the bosonic string, the boundary states are labelled by a Liouville parameter $\sigma$ and matter labels $(k,l)$. In addition to these labels there are also a supercharge parameter $\eta$ related to the linear combination of left and right moving supercharges $G_r + i\eta\tilde{G}_{-r}$ which annihilate the state, and the R-R "charge" $\xi = \pm 1$. Finally, the boundary states also depend on $\zeta = \mathrm{sign}(\mu)$.

Consider first the branes in the positive $\mu$ ($\zeta = +1$) theory. In the expressions that follow, it will be implicit that the matter representation $(k,l)$ and the parameter $\eta$ are correlated by the condition

$$(-1)^{k+l} = \eta \ . \tag{6.1}$$

This follows from the boundary conditions on the supercharge, which imply that the Cardy states with $\eta = 1$ ($\eta = -1$) in are in the NS (R) sector. Using the results of [60,61], we write the boundary states as:

$$\begin{aligned}|\sigma,(k,l);\xi,\eta=+1\rangle\big|_{\zeta=+1} &= \int_0^\infty dP \,\Big( \cos(\pi P\sigma) A_{NS}(P) |P,(k,l);\eta=+1\rangle\rangle_{NS} \\ &\qquad + \xi\cos(\pi P\sigma) A_R(P)|P,(k,l);\eta=+1\rangle\rangle_R \Big) \\ |\sigma,(k,l);\xi,\eta=-1\rangle\big|_{\zeta=+1} &= \int_0^\infty dP \,\Big( \cos(\pi P\sigma) A_{NS}(P) |P,(k,l);\eta=-1\rangle\rangle_{NS} \\ &\qquad - i\xi\sin(\pi P\sigma) A_R(P)|P,(k,l);\eta=-1\rangle\rangle_R \Big) \ ,\end{aligned} \tag{6.2}$$

where the Liouville wavefunctions are given by

$$\begin{aligned} A_{NS}(P) &= \left(\frac{|\mu|}{4}\right)^{iP/b} \frac{\Gamma(1-iPb)\Gamma(1-\frac{iP}{b})}{\sqrt{2}\pi P} \\ A_R(P) &= \left(\frac{|\mu|}{4}\right)^{iP/b} \frac{1}{\pi b^2}\Gamma\left(\frac{1}{2}-iPb\right)\Gamma\left(\frac{1}{2}-\frac{iP}{b}\right) \ ,\end{aligned} \tag{6.3}$$

and the boundary cosmological constant depends on the various parameters as

$$\frac{\mu_B}{\sqrt{|\mu|}} = \begin{cases} \xi \cosh(\frac{\pi b \sigma}{2}) & \eta = +1 \\ \xi \sinh(\frac{\pi b \sigma}{2}) & \eta = -1 \ . \end{cases} \tag{6.4}$$

The factor of $\xi$ in the definition of $\mu_B$ was included for convenience, so as to make explicit the $(-1)^{F_L}$ symmetry of the theory. The action of $(-1)^{F_L}$ maps $\xi \to -\xi$ since it acts as $-1$ $(+1)$ on all R (NS) states. It also maps $\mu_B \to -\mu_B$, since $\mu_B$ is the coefficient of a boundary fermion which transforms like a bulk spin field.

The states $|P,(k,l);\eta\rangle\rangle_{NS,R}$ appearing on the RHS of (6.2) are shorthand for the following linear combination the matter ($M$), Liouville ($L$) and superghost ($G$) Ishibashi states (i.e. strictly, they are not Ishibashi states):

$$\begin{aligned} |P,(k,l);\eta\rangle\rangle_{NS} &= \sum_{k'+l' \text{ even}} \psi_{(\eta,\zeta=+1)}(k,l;k',l')|NS;(k',l');\eta\rangle\rangle_M |NS;P;\eta\rangle\rangle_L |NS;\eta\rangle\rangle_G \\ |P,(k,l);\eta\rangle\rangle_{R} &= \sum_{k'+l' \text{ odd}} \psi_{(\eta,\zeta=+1)}(k,l;k',l')|R;(k',l');\eta\rangle\rangle_M |R;P;\eta\rangle\rangle_L |R;\eta\rangle\rangle_G \ . \end{aligned} \tag{6.5}$$

A few comments are in order.

1. For the superghost states it is convenient to work in the $(-1,-1)$ picture in the NS sector and in the $(-1/2,-3/2)$ or $(-3/2,-1/2)$ in the R sector. In these pictures we see the elementary spacetime fields and the inner products of the states are simplest.

2. The individual Ishibashi states in each sum must have the same label $\eta = \pm 1$. This guarantees that the linear combinations $G_r + i\eta \tilde{G}_{-r}$ annihilate the state, where $G$ and $\tilde{G}$ are the total left and right-moving supercharges of the system. Actually, this would have allowed also a linear combination of Liouville and matter states with opposite $\eta$. Such a linear combination is incompatible with the $P$ dependence in the Cardy state, which is different for the two signs of $\eta$.

3. The restriction in the sums to $k'+l'$ even or $k'+l'$ odd guarantees that we include only NS or R matter Ishibashi states. The "matter wavefunctions" $\psi_{(\eta,\zeta)}(k,l;k',l')$ are closely related to the modular $S$-matrix elements of the combined superminimal model and super-Liouville. Although they can be computed, we will not do that here.

4. It is important to note that the Cardy states (6.2) are not just products of Cardy states of the matter, Liouville and ghost sectors, as they were in the bosonic string. This is because our system is not simply the product of the superminimal model and

super-Liouville theory. Instead, the NS sector is constructed out of the NS sectors of the two theories by projecting on $(-1)^f = 1$, where $f = f_L + f_R$ is the total worldsheet fermion number operator in the combined theory. Similarly, the R sector is made out of the R sectors of the two theories, again by projecting on fixed $(-1)^f$. In the $(-1/2, -3/2)$ or $(-3/2, -1/2)$ picture its value is $-1$ in the 0B theory and $+1$ in the 0A theory.

Let us discuss in some detail the construction of the individual Ishibashi states $|NS, h, \eta\rangle\rangle_{L,M}$ and $|R, h, \eta\rangle\rangle_{L,M}$ in (6.5), where here we will use $h$ to denote the conformal weight of the associated Liouville or matter primary instead of the labels $P$ and $(k, l)$. In the NS sector, the Ishibashi states are linear combinations of NS states of fixed $(-1)^{f_L}$

$$|NS, h, \eta\rangle\rangle_{L,M} = \Big(1 + \eta\, f(h, \widehat{c})\, G^{L,M}_{-1/2} \tilde{G}^{L,M}_{-1/2} + \dots\Big)\, |NS, h\rangle_{L,M} \ , \tag{6.6}$$

where $f(h, \widehat{c})$ is $\eta$-independent and is fixed by the supersymmetry constraints, and $|NS, h\rangle_{L,M}$ denotes the associated primary state. Note that this expression is analogous to the fact that in the Ising model one uses the Ishibashi states $|1\rangle\rangle + \eta|\psi\rangle\rangle$ in forming Cardy states.

In the R sector, every generic (i.e. nonsupersymmetric) representation leads to two Ishibashi states which are annihilated by $G_r + i\eta\tilde{G}_{-r}$ for $\eta = \pm 1$. Consider first the matter or the Liouville part of the theory independently. Then the Ishibashi state in each of them is

$$\begin{aligned} |R, h, \eta\rangle\rangle_{L,M} = & \Big(1 + a(h, \widehat{c})\, L^{L,M}_{-1} \tilde{L}^{L,M}_{-1} + \eta\, b(h, \widehat{c})\, G^{L,M}_{-1} \tilde{G}^{L,M}_{-1}\Big)\, |R, h, \eta\rangle_{L,M} \\ & + \Big(c(h, \widehat{c})\, G^{L,M}_{-1} \tilde{L}^{L,M}_{-1} + \eta\, d(h, \widehat{c})\, L^{L,M}_{-1} \tilde{G}^{L,M}_{-1}\Big)\, |R, h, -\eta\rangle_{L,M} + \dots \end{aligned} \tag{6.7}$$

where again the coefficients appearing here are $\eta$-independent and are determined by the supersymmetry constraints. These Ishibashi states are eigenstates of the fermion number operator in each sector $(-1)^{f^{L,M}}$

$$(-1)^{f^{L,M}} |R, h, \eta\rangle\rangle_{L,M} = -\eta |R, h, \eta\rangle\rangle_{L,M} \ . \tag{6.8}$$

Using (5.11) we conclude that

$$(-1)^f |R, h, \eta\rangle\rangle_L |R, h', \eta\rangle\rangle_M = -|R, h, \eta\rangle\rangle_L |R, h', \eta\rangle\rangle_M \tag{6.9}$$

with generic values of $h$ and $h'$. Therefore, in the 0B theory all the terms in the right hand side of (6.5) made out of these representations satisfy the GSO projection, while no such state survives the projection in the 0A theory.

The situation is somewhat different for the matter R ground state, which exists in the even $(p,q)$ superminimal models and has conformal weight $h_0 = \frac{\widehat{c}_M}{16}$. The matter R ground state must have $(-1)^{f^M} = 1$, but it still leads to two Ishibashi states with $\eta = \pm 1$ and $(-1)^{f^M} = 1$. These take the form

$$|R, h_0, \eta\rangle\rangle_M = \Big(1 + a(h_0, \widehat{c})\, L^M_{-1}\tilde{L}^M_{-1} + \eta\, b(h_0, \widehat{c})\, G^M_{-1}\tilde{G}^M_{-1} + \dots\Big)\, |R, h_0\rangle_M \ , \tag{6.10}$$

where the coefficients are the same as in (6.7). (One can also show that $c(h_0, \widehat{c}) = d(h_0, \widehat{c}) = 0$.) Meanwhile, the Liouville Ishibashi state still has $(-1)^{f^L} = -\eta$, and therefore the contribution of the matter R ground state to (6.5) is restricted by

$$\begin{aligned} &\psi_{(\eta=+1,\zeta=+1)}(k,l;\frac{p}{2},\frac{q}{2}) = 0 \qquad \text{(0B)} \\ &\psi_{(\eta=-1,\zeta=+1)}(k,l;\frac{p}{2},\frac{q}{2}) = 0 \qquad \text{(0A)}\ . \end{aligned} \tag{6.11}$$

Now let us examine the situation for negative $\mu$ ($\zeta = -1$). We start by ignoring the matter and the ghosts. The sign of $\mu$ can be changed in the super-Liouville part of the theory by acting with the $Z_2$ R-transformation $(-1)^{f_L}$. This is not a symmetry of the theory, as it changes the parameter $\mu$. Also, this transformation changes the sign of the projection in the Ramond sector. As for the boundary states, this transformation has the effect of reversing the sign of $\eta$, since it sends $G \to -G$ without changing $\tilde{G}$. Therefore, the boundary super-Liouville theory depends only on $\widehat{\eta} = \eta\zeta$.

Adding back in the matter and superghost sectors, we see that if we want to perform such an R-transformation on the Liouville sector of the theory, we should do it in all the sectors because the total supercharge $G$ is gauged. This is complicated when the matter theory does not have such an R-symmetry. Nevertheless, we can perform such a transformation and label the states by their value of $\widehat{\eta}$. Then our expression (6.2) for the boundary states becomes

$$\begin{aligned} |\sigma,(k,l);\xi,\widehat{\eta}=+1,\zeta\rangle = \int_0^\infty dP\, \Big( &\cos(\pi P\sigma) A_{NS}(P) |P,(k,l);\widehat{\eta}=+1,\zeta\rangle\rangle_{NS} \\ &+ \xi\cos(\pi P\sigma) A_R(P) |P,(k,l);\widehat{\eta}=+1,\zeta\rangle\rangle_R \Big) \\ |\sigma,(k,l);\xi,\widehat{\eta}=-1,\zeta\rangle = \int_0^\infty dP\, \Big( &\cos(\pi P\sigma) A_{NS}(P) |P,(k,l);\widehat{\eta}=-1,\zeta\rangle\rangle_{NS} \\ &- i\xi\sin(\pi P\sigma) A_R(P) |P,(k,l);\widehat{\eta}=-1,\zeta\rangle\rangle_R \Big), \end{aligned} \tag{6.12}$$

where the Liouville wavefunctions are again given by (6.3) and the expression for the boundary cosmological constant (6.4) is generalized to

$$\frac{\mu_B}{\sqrt{|\mu|}} = \begin{cases} \xi \cosh(\frac{\pi b \sigma}{2}) & \widehat{\eta} = +1 \\ \xi \sinh(\frac{\pi b \sigma}{2}) & \widehat{\eta} = -1 \ . \end{cases} \tag{6.13}$$

Now the Cardy states with $\widehat{\eta} = 1$ ($\widehat{\eta} = -1$) in (6.12) are in the NS (R) sector, and therefore their matter representation $(k, l)$ should satisfy

$$(-1)^{k+l} = \widehat{\eta} \ . \tag{6.14}$$

The Ishibashi states (6.5) are generalized to

$$\begin{aligned} |P, (k,l); \widehat{\eta}, \zeta\rangle\rangle_{NS} &= \sum_{k'+l' \text{ even}} \psi_{(\widehat{\eta},\zeta)}(k,l;k',l') |NS; (k',l'); \widehat{\eta}\rangle\rangle_M |NS; P; \widehat{\eta}\rangle\rangle_L |NS; \widehat{\eta}\rangle\rangle_G \\ |P, (k,l); \widehat{\eta}, \zeta\rangle\rangle_{R} &= \sum_{k'+l' \text{ odd}} \psi_{(\widehat{\eta},\zeta)}(k,l;k',l') |R; (k',l'); \widehat{\eta}\rangle\rangle_M |R; P; \widehat{\eta}\rangle\rangle_L |R; \widehat{\eta}\rangle\rangle_G \ . \end{aligned} \tag{6.15}$$

Here the only dependence on $\zeta$ for fixed $\widehat{\eta}$ is through the "matter wavefunctions" $\psi_{(\widehat{\eta},\zeta)}(k,l;k',l')$. This is important in the even $(p,q)$ models which include the Ramond ground state. Here the GSO condition implies

$$\begin{aligned} &\psi_{(\widehat{\eta}=+\zeta,\zeta)}(k,l;\frac{p}{2},\frac{q}{2}) = 0 \qquad \text{(0B)} \\ &\psi_{(\widehat{\eta}=-\zeta,\zeta)}(k,l;\frac{p}{2},\frac{q}{2}) = 0 \qquad \text{(0A)} \ . \end{aligned} \tag{6.16}$$

That is, the Ramond ground state does not contribute to the sum over Ishibashi states for the $\eta = +1$ $(-1)$ brane in the 0B (0A) theory.

To illustrate the general discussion above, let us consider the simplest example of pure supergravity $(p,q) = (2,4)$, which does not have matter at all. (Alternatively, the matter includes the identity and the R-ground state.) Imposing the 0B GSO condition,[7] (6.12) becomes

$$\begin{aligned} |\sigma; \xi, \eta = +1, \zeta = +1\rangle &= \int_0^\infty dP \, \cos(\pi P \sigma) A_{NS}(P) |NS; P; \widehat{\eta}\rangle\rangle_L |NS; \widehat{\eta}\rangle\rangle_G \\ |\sigma; \xi, \eta = -1, \zeta = +1\rangle &= \int_0^\infty dP \, \Big( \cos(\pi P \sigma) A_{NS}(P) |NS; P; \widehat{\eta}\rangle\rangle_L |NS; \widehat{\eta}\rangle\rangle_G \\ &\qquad - i\xi \sin(\pi P \sigma) A_R(P) |R; P; \widehat{\eta}\rangle\rangle_L |R; \widehat{\eta}\rangle\rangle_G \Big) \end{aligned} \tag{6.17}$$

[7] If instead we use the 0A GSO projection, the results below are the same with $\zeta \to -\zeta$ and $\eta \to -\eta$ [24].

for $\mu$ positive. In this phase, the vertex operator for the zero mode of $C$ does not exist, and therefore neither brane is charged. Now consider $\mu$ negative. We have

$$
\begin{aligned}
|\sigma;\xi,\eta=+1,\zeta=-1\rangle &= \int_0^\infty dP\, \cos(\pi P\sigma) A_{NS}(P) |NS;P;\widehat{\eta}\rangle\rangle_L |NS;\widehat{\eta}\rangle\rangle_G \\
|\sigma;\xi,\eta=-1,\zeta=-1\rangle &= \int_0^\infty dP \Big( \cos(\pi P\sigma) A_{NS}(P) |NS;P;\widehat{\eta}\rangle\rangle_L |NS;\widehat{\eta}\rangle\rangle_G \\
&\qquad + \xi \cos(\pi P\sigma) A_R(P) |R;P;\widehat{\eta}\rangle\rangle_L |R;\widehat{\eta}\rangle\rangle_G \Big)\,.
\end{aligned}
\tag{6.18}
$$

In this phase, the $\eta=+1$ brane is again uncharged, but now the $\eta=-1$ brane carries charge. This agrees with the discussion below (5.13), where we saw that for $\mu<0$ we can have charged branes but not flux.

### *6.2. FZZT one-point functions and the role of* $\xi$

Let us use the expression (6.12) for the FZZT boundary state to study the one-point functions of physical operators on the disk with FZZT-type boundary conditions. We expect to find similar results as in the bosonic string, where not all FZZT states are distinct in the BRST cohomology. In particular, we expect (see (3.8)) that all of the FZZT branes with arbitrary matter label can be reduced to elementary branes with fixed matter label, say $(1,1)$ for NS branes and $(1,2)$ or $(2,1)$ for R branes. To actually prove this at the level of the one-point functions as we did for the bosonic string would require knowledge of the matter wavefunctions, which we do not presently have. Thus we will simply assume that it can be done, and from this point onwards we will suppress the matter label and refer to the FZZT branes as $|\,\sigma;\xi,\widehat{\eta},\zeta\rangle$.

Now consider the FZZT one-point functions of the tachyon operators $\mathcal{T}_{r,s}$. From (6.12) we find the $(\sigma,\xi)$-dependent part of these one-point functions can be expressed compactly with the following:

$$
\langle \mathcal{T}_{r,s}|\,\sigma;\xi,\widehat{\eta},\zeta\rangle \propto \xi^{r+s}\cosh\left(\frac{\pi(rq-sp)\sigma}{2\sqrt{pq}}+\frac{i\pi(r+s)\widehat{\nu}}{2}\right) \tag{6.19}
$$

where we have defined

$$
\widehat{\nu}=\frac{1-\widehat{\eta}}{2}=\begin{cases}0 & \text{for } \widehat{\eta}=+1\\ 1 & \text{for } \widehat{\eta}=-1\end{cases} \tag{6.20}
$$

The one-point functions of the other physical operators are also given by (6.19), but with different values of $s$. Thus physical one-point functions are invariant under the transformations

$$
(\sigma,\ \xi)\to(-\sigma,\ \widehat{\eta}\,\xi),\qquad (\sigma\pm 2i\sqrt{pq},\ (-1)^p\xi) \tag{6.21}
$$

Note that the first transformation is true at the level of the boundary state (6.12). The second transformation is only true for physical one-point functions, which is evidence that FZZT states related by this transformation differ by a BRST exact state. These transformations suggest that we define

$$z = \begin{cases} \cosh\left(\frac{\pi\sigma}{2\sqrt{pq}} - \frac{i\pi\widehat{\nu}}{2}\right) & (p,q) \text{ odd} \\ \cosh\left(\frac{\pi\sigma}{\sqrt{pq}} - \frac{i\pi\widehat{\nu}}{2}\right) & (p,q) \text{ even} \end{cases} \tag{6.22}$$

and label the FZZT brane by $z$:

$$|\,\sigma;\xi,\widehat{\eta},\zeta\rangle \to |\,z;\xi,\widehat{\eta},\zeta\rangle \tag{6.23}$$

so that at *fixed* $\xi$, two FZZT branes labelled by $z$ and $z'$ are equal if and only if $z = z'$.

The parametrization in terms of $z$ eliminates some, but not all of the redundancy of description implied by the transformations (6.21). The remaining redundancy involves changing the sign of $\xi$. Indeed, we see that when $\widehat{\eta} = -1$ or $(p,q)$ is odd, a state with $(z,-\xi)$ is equivalent to a state with $(-z,\xi)$. For these states, $\xi$ is a redundant label which can be removed by analytic continuation in $z$. The only states for which $\xi$ cannot be eliminated in this way are those with $\widehat{\eta} = +1$ when $(p,q)$ is even.[8] In section 7, we will see how these disparate facts can be all understood geometrically and in a unified way, in terms of an auxiliary Riemann surface.

Finally, we note that when $(p,q)$ is odd, the transformation

$$(\sigma,\ \widehat{\eta}) \to (i\sqrt{pq} - \sigma,\ -\widehat{\eta}) \tag{6.24}$$

leaves the $\sigma$-dependent part of the one-point functions (6.19) unchanged. Moreover, the transformation also leaves $z$ unchanged. This suggests that when $(p,q)$ is odd, the FZZT states with $\widehat{\eta} = -1$ are equivalent to the FZZT states with $\widehat{\eta} = +1$ (but with different $\sigma$) in the BRST cohomology. Let us assume that this is true, and focus our attention only on the states with $\widehat{\eta} = +1$ when $(p,q)$ is odd. We will also motivate this simplification geometrically in section 7 and we will use it in section 8.

[8] We could further reparametrize the FZZT branes so as to remove the redundancy by $\xi$. However, as this would needlessly complicate the notation, we prefer to continue labelling the FZZT branes with $\xi$, even in the cases where it is unnecessary.

*6.3. ZZ branes and their one-point functions*

The discussion of the ZZ boundary states is analogous to that of the FZZT states, so we will simply write down the relevant expressions. Here we will not attempt to analyze the subtleties of the degenerate super-Virasoro representations that arise at rational $b^2$. For the bosonic string, we found that these subtleties only pertained to $(m,n)$ ZZ branes with $m > p$ or $n > q$. The ZZ boundary states with $m \le p$ and $n \le q$ were given by the formula for generic $b$, and could be written as a difference of just two FZZT branes. Therefore the subtleties at rational $b^2$ did not affect our final conclusion in the bosonic string, which was that the set of all $(m,n)$ ZZ branes could be reduced to a principal set with $m < p$ and $n < q$ and $mq - np > 0$. As we expect a similar conclusion in the superstring, let us restrict our attention from the outset to ZZ boundary states with $m \le p$ and $n \le q$. Then by analogy with the bosonic string, these should be given by the formula at generic $b$ [60,61]:

$$
\begin{aligned}
&|(m,n),(k,l);\xi,\widehat{\eta}=+1,\zeta\rangle = \\
&\quad 2\int_0^\infty dP\,\Bigg(\sinh\big(\frac{\pi P\,m}{b}\big)\sinh\big(\pi P\,b\,n\big)A_{NS}(P)|P,(k,l);\eta,\zeta\rangle\rangle_{NS} \\
&\qquad\qquad + \xi\sinh\big(\frac{\pi m P}{b}+\frac{i\pi n}{2}\big)\sinh\big(\pi n P\,b-\frac{i\pi n}{2}\big)A_R(P)|P,(k,l);\eta,\zeta\rangle\rangle_R\Bigg) \\
&|(m,n),(k,l);\xi,\widehat{\eta}=-1,\zeta\rangle = \\
&\quad 2\int_0^\infty dP\,\Bigg(\sinh\big(\frac{\pi P\,m}{b}\big)\sinh\big(\pi P\,b\,n\big)A_{NS}(P)|P,(k,l);\eta,\zeta\rangle\rangle_{NS} \\
&\qquad\qquad + \xi\cosh\big(\frac{\pi m P}{b}+\frac{i\pi n}{2}\big)\sinh\big(\pi n P\,b-\frac{i\pi n}{2}\big)A_R(P)|P,(k,l);\eta,\zeta\rangle\rangle_R\Bigg)\ .
\end{aligned}
\tag{6.25}
$$

As for the FZZT branes, the parameter $\widehat{\eta}$ determines whether the Cardy state is NS or R. Thus we must require

$$
(-1)^{m+n} = (-1)^{k+l} = \widehat{\eta}\ . \tag{6.26}
$$

Also, we will assume as we did for the FZZT branes that the ZZ branes with different matter labels can be reduced down to ZZ branes with fixed matter label.

Expanding the products of cosh and sinh using standard trigonometric identities and comparing with (6.12), we see that the ZZ branes with $m \le p$ and $n \le q$ can be written in terms of the FZZT branes as (we suppress the labels $(\widehat{\eta},\zeta)$):

$$
|\,m,n;\xi\rangle = |\,z=z(m,n);\xi\rangle - |\,z=z(m,-n);(-1)^n\xi\rangle \tag{6.27}
$$

where $z(m,n) \equiv z(\sigma(m,n))$ was defined in (6.22), and

$$\sigma(m,n) = i\left(\frac{m}{b} + nb\right) \ . \tag{6.28}$$

The boundary cosmological constant corresponding to $\sigma(m,n)$ is

$$\mu_B(m,n,\xi) = \mu_B(m,-n,(-1)^n\xi) = \begin{cases} \xi\sqrt{|\mu|}\cos\frac{\pi}{2}(m+nb^2) & \widehat{\eta}=+1 \\ i\,\xi\sqrt{|\mu|}\sin\frac{\pi}{2}(m+nb^2) & \widehat{\eta}=-1\ . \end{cases} \tag{6.29}$$

Note that the two FZZT branes in the right hand side of (6.27) always have the same value of $\mu_B$.

As in the bosonic string, we do not expect all of the ZZ branes to be distinct in the full string theory; rather, we expect many to differ by BRST null states. Indeed, the formula (6.27) for the ZZ branes immediately implies the identification

$$|p-m, q-n; (-1)^{q+m+n}\xi\rangle = |m,n;\xi\rangle \tag{6.30}$$

modulo BRST null states. Moreover, a straightforward computation using (6.25) shows that the one-point functions of physical operators all vanish when $m=p$ or $n=q$. This suggests that in the BRST cohomology:

$$|m,n;\xi\rangle = 0, \qquad \text{when} \ \ m=p \ \ \text{or} \ \ n=q\ . \tag{6.31}$$

In the next section, we will also motivate these identifications from a more geometrical point of view, as was done for the bosonic string.

Using the identifications (6.30) and (6.31), we can reduce the infinite set of $(m,n,\xi)$ ZZ branes down to what we will call, as in the bosonic string, the principal ZZ branes. Let us define $\mathcal{B} = \{(m,n)|\ m=1,\dots,p-1,\ n=1,\dots,q-1\}$. Then for $(p,q)$ even, the principal ZZ branes are

$$\begin{aligned} &\widehat{\eta}=+1:\ (m,n,\xi)\ , \quad (m,n)\in\mathcal{B}\ , \quad m+n \ \text{even}\ , \quad mq-np\geq 0\ , \quad \xi=\pm1 \\ &\widehat{\eta}=-1:\ (m,n,\xi)\ , \quad (m,n)\in\mathcal{B}\ , \quad m+n \ \text{odd}\ , \quad \xi=+1\ , \end{aligned} \tag{6.32}$$

giving a total of $\frac{(p-1)(q-1)\pm1}{2}$ principal ZZ branes for $\widehat{\eta}=\pm1$. On the other hand, for $(p,q)$ odd, the principal ZZ branes are

$$(m,n,\xi)\ , \quad (m,n)\in\mathcal{B}\ , \quad m+n \ \text{even}\ , \quad \xi=+1 \qquad (\text{and}\ \widehat{\eta}=+1) \tag{6.33}$$

giving a total of $\frac{(p-1)(q-1)}{2}$ principal ZZ branes for $(p,q)$ odd. Notice that we have restricted ourselves to the ZZ branes with $\widehat{\eta}=+1$. Since the map (6.24) $(\sigma,\ \widehat{\eta})\to(i\sqrt{pq}-\sigma,\ -\widehat{\eta})$ leaves $z$ unchanged, it preserves the formula (6.27) and maps ZZ branes with $\widehat{\eta}$ to ZZ branes with $-\widehat{\eta}$.

In the bosonic string, we saw that we could normalize the ground ring elements so that the principal ZZ branes became eigenstates of the ground ring. We expect similar phenomena to occur in the superstring. To make more precise statements, we would need to have explicit formulas for the matter wavefunctions. We will leave this for future work.

## 7. Geometric interpretation of minimal superstring theory

### 7.1. The surfaces $\mathcal{M}^{\pm}_{p,q}$ and their analytic structure

As in the bosonic string, we can understand many of the features of the boundary states using an auxiliary Riemann surface that emerges from the FZZT partition function. We start as before with the disk one-point function of the cosmological constant operator. This is given by:

$$\partial_\mu Z\big|_{\mu_B}=\langle\psi\overline{\psi}e^{b\phi}\rangle=A(b)(\sqrt{|\,\mu|})^{1/b^2-1}\cosh\left(b-\frac{1}{b}\right)\frac{\pi\sigma}{2}\ . \tag{7.1}$$

Notice the similarity to the bosonic one-point function (4.2). The difference comes in the relation between $\sigma$ and $\mu_B$ for $\widehat{\eta}=-1$. For simplicity, we will limit our discussion in the superstring to the FZZT brane and not its dual. The relationship between the FZZT brane and its dual is exactly analogous to that in the bosonic string.

Integrating (7.1) leads to

$$Z=C(b)(\sqrt{|\mu|})^{1/b^2+1}\times\begin{cases} b^2\cosh\left(\frac{\pi\sigma}{2b}\right)\cosh\left(\frac{\pi b\sigma}{2}\right)-\sinh\left(\frac{\pi\sigma}{2b}\right)\sinh\left(\frac{\pi b\sigma}{2}\right) & \widehat{\eta}=+1\\ b^2\sinh\left(\frac{\pi\sigma}{2b}\right)\sinh\left(\frac{\pi b\sigma}{2}\right)-\cosh\left(\frac{\pi\sigma}{2b}\right)\cosh\left(\frac{\pi b\sigma}{2}\right) & \widehat{\eta}=-1\ ,\end{cases} \tag{7.2}$$

and differentiating this with respect to $\mu_B$, we find

$$\partial_{\mu_B}Z\big|_{\mu}=\begin{cases}(\sqrt{|\mu|})^{1/b^2}\xi\cosh\left(\frac{\pi\sigma}{2b}\right) & \widehat{\eta}=+1\\ (\sqrt{|\mu|})^{1/b^2}\xi\sinh\left(\frac{\pi\sigma}{2b}\right) & \widehat{\eta}=-1\ .\end{cases} \tag{7.3}$$

In (7.1) and (7.2) above, $A(b)$ and $C(b)$ are overall normalization factors that were chosen so as to make the normalization of (7.3) unity. As in the bosonic string, we suppress the contribution of the matter sector, since this is $\mu$ and $\mu_B$ independent. The two types of

FZZT brane lead to two Riemann surfaces; we will call them $\mathcal{M}^{\widehat{\eta}}_{p,q}$. For reasons that will shortly become clear, we will parametrize these surfaces with slightly different dimensionless coordinates:

$$\begin{aligned} \mathcal{M}^{+}_{p,q}: &\quad x = \frac{\mu_B}{\sqrt{\mu}} \ , \quad y = \frac{\partial_{\mu_B} Z}{\sqrt{\tilde{\mu}}} \ , \\ \mathcal{M}^{-}_{p,q}: &\quad x = \frac{i\mu_B}{\sqrt{\mu}} \ , \quad y = \frac{i\partial_{\mu_B} Z}{\sqrt{\tilde{\mu}}} \ . \end{aligned} \tag{7.4}$$

Then in terms of $x$ and $y$, the surfaces are described by the polynomial equations:

$$0 = F(x,y) = \begin{cases} T_q(x) - T_p(y) & \widehat{\eta} = +1 \\ (-1)^{\frac{p-q}{2}} T_q(x) - T_p(y) & \widehat{\eta} = -1 \ . \end{cases} \tag{7.5}$$

Note that we have assumed in writing (7.5) that $(p,q)$ are either both odd or both even; otherwise there could be additional phases arising from the factor of $\xi$ in the definition of the coordinates. It is interesting that for $(p,q)$ odd, the phase in front of $T_q$ in the second line of (7.5) can be absorbed into the argument of the Chebyshev polynomial. Thus, for $(p,q)$ odd the surfaces $\mathcal{M}^{+}_{p,q}$ and $\mathcal{M}^{-}_{p,q}$ are identical, and we will limit our discussion to $\mathcal{M}^{+}_{p,q}$ without loss of generality. However, for $(p,q)$ even, since $(p-q)/2$ is odd the phase cannot be absorbed, and therefore these surfaces are different. This is consistent with the suggestion that for $(p,q)$ odd, boundary states with $\widehat{\eta} = \pm 1$ become equivalent, while for $(p,q)$ even, $\widehat{\eta}$ labels distinct states.

As in the bosonic string, a natural uniformization of these surfaces is provided by the parameter $z$ defined in (6.22). The coordinates of $\mathcal{M}^{\widehat{\eta}}_{p,q}$ are given in terms of $z$ by

$$(x,y) = \begin{cases} \Big(\xi T_p(z), \xi T_q(z)\Big) & (p,q) \text{ odd} \\ \Big(\xi T_{\frac{p}{2}}(z), \xi T_{\frac{q}{2}}(z)\Big) & (p,q) \text{ even and } \widehat{\eta} = +1 \\ \Big(\xi z U_{\frac{p}{2}-1}(\sqrt{1-z^2}), \xi z U_{\frac{q}{2}-1}(\sqrt{1-z^2})\Big) & (p,q) \text{ even and } \widehat{\eta} = -1 \end{cases} \tag{7.6}$$

Therefore, for fixed $\xi$, $z \in \mathbb{C}$ covers $\mathcal{M}^{\widehat{\eta}}_{p,q}$ exactly once, except for when $(p,q)$ is even and $\widehat{\eta} = -1$, where we must extend $z$ to a two-sheeted cover of the complex plane in order to cover the surface once. Notice also that $\xi$ can be absorbed into the definition of $z$ except when $(p,q)$ is even and $\widehat{\eta} = +1$. (Recall that for the $(p,q)$ even models, either $p/2$ or $q/2$ must be even.) This gives a nice geometric interpretation to the results of section 6, where we saw that only for $(p,q)$ even and $\widehat{\eta} = 1$ did the parameter $\xi$ label two distinct FZZT boundary states. The fact that $\xi$ cannot be eliminated in this case also has implications for the analytic structure of the associated surface that we will discuss below.

It is straightforward to work out the analytic structure of $\mathcal{M}^+_{p,q}$ and $\mathcal{M}^-_{p,q}$. One can use either the condition $F = dF = 0$, or equivalently one can look for points $(x, y)$ that correspond to multiple values of $z$. Either way, we must consider the following three cases:

1. *$(p, q)$ odd and coprime ($\widehat{\eta} = +1$)*
   For $(p, q)$ odd, $\mathcal{M}^+_{p,q}$ is identical to the bosonic surface $\mathcal{M}_{p,q}$ (figure 1). Thus it has the $(p-1)(q-1)/2$ singularities given by (4.14), which we include here for the sake of completeness:

$$(x, y) = \left(\cos\frac{\pi(jq+kp)}{q}\ , \cos\frac{\pi(jq+kp)}{p}\right)\ , \qquad 1 \le j \le p-1,\ 1 \le k \le q-1,\ j\,q - k\,p > 0 \tag{7.7}$$

2. *$(p, q)$ even, $(\frac{p}{2}, \frac{q}{2})$ coprime, $\frac{p-q}{2}$ odd, $\widehat{\eta} = +1$*
   For $(p, q)$ even, the surface $\mathcal{M}^+_{p,q}$ is actually quite special. Since its curve can be written as $T_{\frac{q}{2}}(x)^2 = T_{\frac{p}{2}}(y)^2$, the surface splits into two separate branches described by $T_{\frac{q}{2}}(x) = \pm T_{\frac{p}{2}}(y)$, with each branch itself a Riemann surface. We will denote these surfaces by $\left(\mathcal{M}^+_{p,q}\right)_\pm$. Note that since either $\frac{p}{2}$ or $\frac{q}{2}$ must be odd, the two branches of the surface are related by the map $(x, y) \to (-x, -y)$. From the definition (7.4) of $x$ and $y$, we see that this is just the action of $\xi \to -\xi$. Thus we can think of the two branches as $\left(\mathcal{M}^+_{p,q}\right)_\xi$. We will elaborate on this identification of the branches with $\xi$ shortly.

   One can easily check that $\mathcal{M}^+_{p,q}$ has $\frac{(p-1)(q-1)+1}{2}$ singularities described by

$$(x, y) = \left(\cos\frac{j\pi}{q}, \cos\frac{k\pi}{p}\right),\ j = 1, \ldots, q-1,\ k = 1\ \ldots, p-1,\ k - j = 0 \bmod 2. \tag{7.8}$$

   There are two types of singularity: those that are singularities of $\left(\mathcal{M}^+_{p,q}\right)_\xi$ individually, which we will call *regular singularities*; and those that join the two branches, which we will call *connecting singularities*. The distinction between the two is quite simple algebraically: connecting singularities satisfy $T_{\frac{q}{2}}(x) = T_{\frac{p}{2}}(y) = 0$, while the regular singularities have $T_{\frac{q}{2}}(x)$ and $T_{\frac{p}{2}}(y) \neq 0$. Counting the number of singularities of each type is also easy. There are $\frac{p\,q}{4}$ connecting singularities; and $\frac{(p-2)(q-2)}{8}$ regular singularities on each branch of $\mathcal{M}^+_{p,q}$. An example of this kind of surface is shown in figure 2.

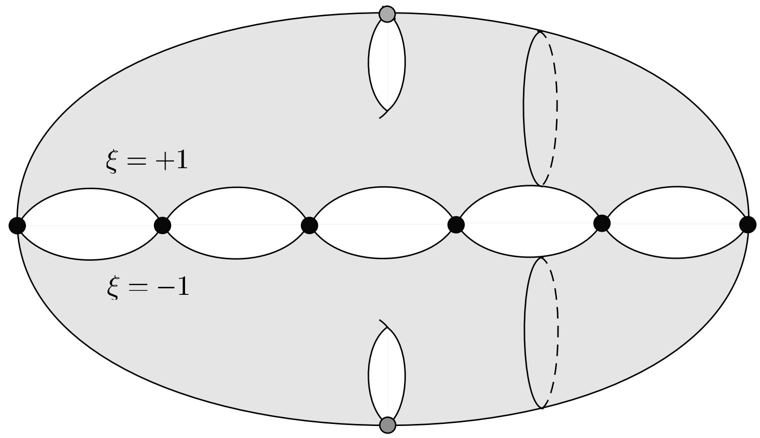


**Fig. 2:** The surface $\mathcal{M}^+_{p,q}$ for $(p,q)$ even, shown here for $(p,q) = (4,6)$. The two subsurfaces of $\mathcal{M}^+_{p,q}$ are labelled by $\xi = \pm 1$. The black points represent the connecting singularities that join the two subsurfaces, while the gray points represent the regular singularities of each subsurface.

3. *$(p,q)$ even, $(\frac{p}{2},\frac{q}{2})$ coprime, $\frac{p-q}{2}$ odd, $\widehat{\eta}=-1$*
   Unlike the previous case, here $\mathcal{M}^-_{p,q}$ does not split into two branches, since the equation for this surface takes the form $T_{\frac{p}{2}}(y)^2 + T_{\frac{q}{2}}(x)^2 = 1$. One can show that $\mathcal{M}^-_{p,q}$ has $\frac{(p-1)(q-1)-1}{2}$ singularities located at

$$(x,y) = \left(\cos\frac{j\pi}{q}, \cos\frac{k\pi}{p}\right), \; j=1,\ldots,q-1, \; k=1\ldots,p-1, \; k-j=1 \bmod 2. \tag{7.9}$$

To summarize: in the four cases we considered above, the only surface that split into two separate branches was $\mathcal{M}^+_{p,q}$ with $(p,q)$ even. Moreover, we were able to identify these two branches with the two signs of $\xi$.

Having worked out the singularities of our surfaces, let us now see that they match the locations of the principal ZZ branes, as was the case in the bosonic string. The principal ZZ branes have boundary cosmological constant given by (6.29). Using (7.3), we find that the $\widehat{\eta}=+1$ branes are located at the points

$$(x,y) = \left(\xi\cos\frac{\pi}{2q}\big(mq+np\big), \xi\cos\frac{\pi}{2p}\big(mq+np\big)\right) \tag{7.10}$$

in $\mathcal{M}^+_{p,q}$, with $(m,n,\xi)$ taking values appropriate to the principal $\widehat{\eta}=+1$ ZZ branes as described in (6.32) and (6.33). Similarly, the $\widehat{\eta}=-1$ branes are located at

$$(x,y) = \left(\xi\cos\frac{\pi}{2q}\big((m+1)q+np\big), \xi\cos\frac{\pi}{2p}\big(mq+(n+1)p\big)\right) \tag{7.11}$$

in $\mathcal{M}^-_{p,q}$, with $(m,n,\xi)$ again taking the appropriate values. We claim that the locations of the principal ZZ branes are just a different parametrization of the singularities of $\mathcal{M}^\pm_{p,q}$. Let us briefly sketch a proof. First, one can show that they indeed lie at the singularities by checking that their locations $(x,y)$ satisfy the conditions $F(x,y)=0$ and $dF(x,y)=0$. Since the locations of the principal ZZ branes are all distinct, and there are exactly as many principal $\widehat{\eta}$ ZZ branes as there are singularities of $\mathcal{M}^{\widehat{\eta}}_{p,q}$, it follows that their locations match the singularities exactly.

Combining the calculations of the one-point functions in the previous section with the geometric results here, we can summarize our conclusions regarding the ZZ branes very succintly. These conclusions are essentially the same as in the bosonic string. The $\widehat{\eta}=\pm 1$ principal ZZ branes are located at the singularities of our surfaces $\mathcal{M}^\pm_{p,q}$. ZZ branes located at the same singularity of the surface differ by BRST null states, while ZZ branes located at different singularities are distinct BRST cohomology classes. Finally, ZZ branes that are not located at singularities of the surface are themselves BRST null.

By analogy with the bosonic string, we also expect that the $\widehat{\eta}$ FZZT and ZZ branes can be represented as contour integrals of a one-form (or boundary state) on the surfaces $\mathcal{M}^{\widehat{\eta}}_{p,q}$. For instance, in the cases where $\xi$ is a redundant parameter, the $\widehat{\eta}$ FZZT branes are given by line integrals of the one-form $y\,dx$ on $\mathcal{M}^{\widehat{\eta}}_{p,q}$:

$$Z(\mu_B)=\mu^{\frac{p+q}{2p}}\int_{\mathcal{P}}^{x(\mu_B)} y\,dx\ , \tag{7.12}$$

where $\mathcal{P}$ is an arbitrary fixed point in $\mathcal{M}^{\widehat{\eta}}_{p,q}$. For the case of $(p,q)$ even and $\widehat{\eta}=+1$ where $\xi$ actually labels two distinct branes, we saw above that the corresponding surface splits into two separate branches. These branches were also labelled by $\xi$, and they were connected by singularities. Thus in this case alone can we define two inequivalent line integrals on the surface; these clearly correspond to the two types of FZZT brane:

$$Z(\mu_B,\xi)=\mu^{\frac{p+q}{2p}}\int_{\mathcal{P}_\xi}^{x(\mu_B)} y\,dx\ . \tag{7.13}$$

Here $\mathcal{P}_\xi\in\left(\mathcal{M}^+_{p,q}\right)_\xi$ is an arbitrary fixed point on the branch labelled by $\xi$.

We can consider the ZZ branes in a similar way. When $\xi$ is a redundant parameter, the relation (6.27) between the ZZ and FZZT branes, together with (7.12), implies that

the ZZ branes can be written as closed contour integrals of $y\,dx$. As in the bosonic string, we can promote this to a relation between boundary states:

$$|m,n;\widehat{\eta}\rangle = \oint_{B_{m,n}} \partial_x|z(x);\widehat{\eta}\rangle \; dx \ , \tag{7.14}$$

where the contour $B_{m,n}$ runs through the open cycle on $\mathcal{M}^{\pm}_{p,q}$ and the singularity associated to the $(m,n)$ ZZ brane. For $\widehat{\eta}=+1$ and $(p,q)$ odd, we saw that $\xi$ was only redundant modulo BRST exact states. Thus in this case, we must interpret (7.14) as a relation in the BRST cohomology of the full theory.

Once again, the case of $(p,q)$ even and $\widehat{\eta}=+1$ has more structure. The relation (6.27) implies that for $n$ even, the ZZ branes are differences of FZZT branes of the same $\xi$. Using (7.13), we see that for $n$ even, we can write the ZZ brane as a closed contour of the FZZT brane:

$$|m,n;\xi,\widehat{\eta}=1\rangle\big|_{(m,n)\text{ even}} = \oint_{B^{\xi}_{m,n}} \partial_x|z(x);\xi,\widehat{\eta}=1\rangle \; dx \ , \tag{7.15}$$

where the contour $B^{\xi}_{m,n}$ runs through the open cycle on $\left(\mathcal{M}^{+}_{p,q}\right)_{\xi}$ and the singularity associated to the $(m,n,\xi)$ ZZ brane. On the other hand, for $n$ odd, the relation (6.27) says that the ZZ branes are differences of FZZT branes of opposite $\xi$. Thus these ZZ branes are not described by closed contours, but rather by line integrals from one branch of $\mathcal{M}^{+}_{p,q}$ to another:

$$|m,n;\xi,\widehat{\eta}=1\rangle\big|_{(m,n)\text{ odd}} = \int_{\mathcal{P}_{\xi}}^{x_{m,n}} \partial_x|z(x),\xi,\widehat{\eta}=1\rangle \; dx - \int_{\mathcal{P}_{-\xi}}^{x_{m,n}} \partial_x|z(x),-\xi,\widehat{\eta}=1\rangle \; dx \ . \tag{7.16}$$

Note that in order for these relations to be true, the properties of the singularity associated to the $(m,n,\xi)$ ZZ brane must depend on $n$ in a very non-trivial way: for $n$ even, the singularity must be a regular singularity of the branch $\left(\mathcal{M}^{+}_{p,q}\right)_{\xi}$, while for $n$ odd, it must be a connecting singularity between the two branches. (See the discussion following (7.8) for a description of the two types of singularities.) Substituting the locations of the ZZ branes into $T_{\frac{q}{2}}(x)$ and $T_{\frac{p}{2}}(y)$, we immediately find that these polynomials are zero for the $n$ odd branes and nonzero for the $n$ even branes. Therefore the $n$ odd (even) branes are indeed located at the connecting (regular) singularities, which is a highly non-trivial check of our relations (7.15) and (7.16). Figure 3 illustrates the different types of FZZT and ZZ branes for this surface.

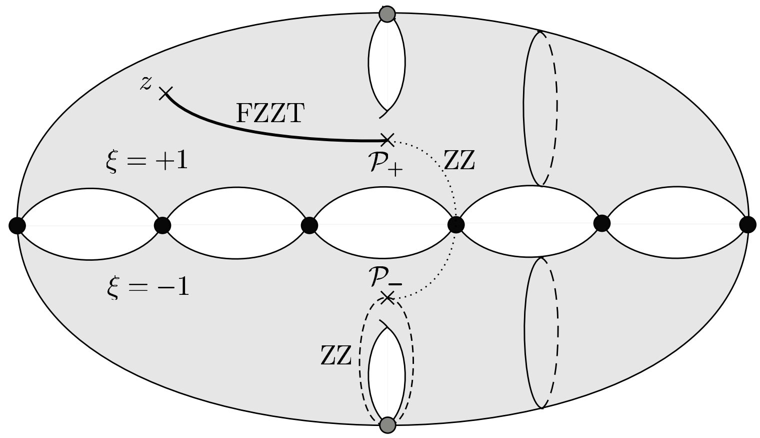


**Fig. 3:** The surface $\mathcal{M}^+_{p,q}$ for $(p,q)$ even, along with examples of FZZT and ZZ brane contours, shown here again for $(p,q) = (4,6)$. There are now two types of FZZT brane labelled by $\xi$ (only one is drawn in the figure), and these are given by contours (solid) from the base point $\mathcal{P}_\xi$ on the subsurface $(\mathcal{M}^+_{p,q})_\xi$ to a point $z$ in $\mathcal{M}^+_{p,q}$. There are also two classes of ZZ branes: the neutral branes, which are described by a closed contour (dashed) through a regular singularity; and the charged branes, which are described by an open contour (dotted) that runs from $\mathcal{P}_+$ to $\mathcal{P}_-$ through a connecting singularity.

It follows immediately from these relations that the ZZ branes with $n$ even are neutral branes, being differences of FZZT branes of the same charge. Meanwhile the ZZ branes with $n$ odd are differences of two FZZT branes with opposite values of $\xi$, so they are charged. We could have also seen this directly from the behavior of Ramond part of the ZZ boundary state (6.25) at zero momentum. Note that the connecting singularities come in pairs $(x,y)$ and $(-x,-y)$. These correspond to a charged ZZ brane and its antibrane.

*7.2. Deformations of $\mathcal{M}^\pm_{p,q}$*

Just as in the bosonic string, we can consider deformations of the surfaces $\mathcal{M}^\pm_{p,q}$ that preserve the number of singularities on each surface. For $(p,q)$ odd, there is really only one surface, and it is identical to the surface $\mathcal{M}_{p,q}$ of the bosonic string. Therefore the discussion of deformations in section 4.2 applies equally well here. This agrees with the fact that the spectrum of physical operators in the $(p,q)$ odd bosonic and supersymmetric minimal string theories are identical in every respect, including their KPZ scalings. This also agrees with the fact that the $(p,q)$ odd minimal superstring theory is the same at $\mu > 0$ and at $\mu < 0$.

For $(p,q)$ even, there are two distinct surfaces $\mathcal{M}^\pm_{p,q}$ described by the equations $F_\pm(x,y) \equiv T_p(y) \mp T_q(x) = 0$. The number of singularities of the two surfaces is

$\frac{(p-1)(q-1)\pm 1}{2}$. By arguments analogous to those in section 4.2, we find that a complete and independent basis for the deformations is given by

$$\begin{aligned} &\delta_{r,s}F_{\pm}(x,y) = \tilde{t}_{r,s}\Big(U_{q-s-1}(x)U_{r-1}(y) \mp U_{p-r-1}(y)U_{s-1}(x)\Big) \\ &1 \le r \le p\ , \quad s \le q-1\ , \quad qr - ps \ge 0\ . \end{aligned} \tag{7.17}$$

Note the one essential difference between the deformations of $\mathcal{M}^{\pm}_{p,q}$: for $\mathcal{M}^{+}_{p,q}$, the deformation with $(r,s) = (\frac{p}{2},\frac{q}{2})$ is not present, because the two terms in (7.17) are equal and cancel. In other words, there is no singularity preserving deformation of $\mathcal{M}^{+}_{p,q}$ that corresponds to the Ramond ground state.

For the $\eta = -1$ brane, this is precisely what is needed. For $\mu > 0$, the brane is described by $\mathcal{M}^{-}_{p,q}$, which does have a deformation corresponding to the Ramond ground state. Meanwhile for $\mu < 0$, the brane is described by $\mathcal{M}^{+}_{p,q}$, which does not have a deformation corresponding to the Ramond ground state. This agrees with our study of the 0B spectrum in section 5, where we saw that the Ramond ground state in the $(-1/2,-1/2)$ picture was present for $\mu > 0$ and not present for $\mu < 0$.

Consider now the $\eta = +1$ brane. For $\mu > 0$ there is no problem, because although the Ramond ground state exists in the string theory, its deformation of the surface is zero. This occurs for two reasons: the Liouville part of the tachyon one-point function vanishes for the Ramond ground state, and the matter wavefunction also vanishes by (6.16). However, for $\mu < 0$ the surface $\mathcal{M}^{-}_{p,q}$ has a deformation with $(r,s) = (\frac{p}{2},\frac{q}{2})$, but the Ramond ground state does not exist in the $(-1/2,-1/2)$ picture. We do not know what this deformation corresponds to.

Finally, we can extend the bosonic string discussion of the effect a background with many $(m,n)$ ZZ branes to the superstring. Adding to the system $N_{m,n} \sim 1/g_s$ ZZ branes the singularity associated with the pinched cycle $A_{m,n}$ is opened up and as in (4.32),

$$\oint_{A_{m,n}} y\,dx = g_s N_{m,n}\ . \tag{7.18}$$

When the $(m,n)$ D-brane is charged, this has the effect of adding flux to the system which can be measured by an expression similar to (7.18). Indeed, in [24] the flux due to charged D-branes was defined by such a contour integral, and its effect on the dynamics was discussed.

## 8. Comparing the $(p, q)$ odd supersymmetric and bosonic models

In this section, we will compare the bosonic and 0B supersymmetric minimal string theories with $(p, q)$ odd.

First, the two theories have the same spectrum of physical operators [24,33]. In particular, they both have $(p-1)(q-1)/2$ tachyons, and their ground rings are isomorphic. Moreover, since the $\mu$-deformed ground ring multiplication is the same in both theories, the arguments of section 2.2 and 5.2 imply that the tachyon $N$-point functions for $N \leq 3$ are the same in the two theories. Both will be given essentially by the fusion rules in the minimal model, which are the same as the fusion rules of the superminimal model. A more interesting test of our conjecture would be to compare the general ($N > 4$) correlation functions of the two theories. This might probe structure of the theories that goes beyond their ground rings.

Second, as we saw in section 5, the $(p, q)$ odd minimal superstring theories do not have the Ramond ground state in their spectrum, for either sign of $\mu$. Thus for these models, worldsheet supersymmetry is always broken. Moreover, since the Ramond ground state measures RR charge, it is not possible to define such a charge for the $(p, q)$ odd minimal superstring. These are of course both necessary conditions if the supersymmetric and bosonic theories are to be the same.

The boundary states of the bosonic and supersymmetric models offer more opportunities for comparison. As we saw in section 7.1, they are described by the same Riemann surface $\mathcal{M}_{p,q}$. Thus it is reasonable to expect the bosonic and supersymmetric theories to have an identical set of FZZT and ZZ branes. We will not offer a detailed comparison, but will instead focus on the simplest test: counting the total number of such states. In the bosonic string, we found that the set of all FZZT branes reduced to a one-parameter family (parametrized by $\sigma$ or $z$) of FZZT branes with matter state $(1, 1)$. We also found exactly $(p-1)(q-1)/2$ independent ZZ branes distinguished by their Liouville label $(m, n)$.

Compare this with the boundary states in the superstring. These states are in general labelled by $\xi = \pm 1$ and $\widehat{\eta} = \pm 1$ in addition to their Liouville and matter labels. For $(p, q)$ odd, we suggested in section 6.2 that both $\xi$ and $\widehat{\eta}$ are redundant parameters. These arguments were also motivated geometrically in section 7. Then assuming that there is a reduction of the FZZT branes with general matter state to the FZZT branes with $(1, 1)$ matter state, as in the bosonic string, we find the same one-parameter family of independent FZZT branes. We also find exactly $(p-1)(q-1)/2$ ZZ branes, as in the

bosonic string, again assuming the reduction in matter states. Clearly, it would be nice to justify this assumption with an explicit computation. For this, we would need explicit expressions for the matter wavefunctions.

We will see in the next section that the Riemann surface is precisely the surface that defines the dual matrix model description. Therefore the bosonic and supersymmetric $(p, q)$ models have the same matrix model, and we expect them to agree to all orders in string perturbation theory.

We will conclude this section with a list of further tests of our proposed duality that should be carried out. The 0B superstring has a $\mathbb{Z}_2$ symmetry and orbifolding by it leads to the 0A theory. It is not known whether an analogous construction exists in the bosonic string. It also remains to understand what it means to have negative $\mu$ in the bosonic string, since the superstring clearly exists for both signs of $\mu$, and moreover it is invariant under $\mu \to -\mu$. Finally, a more detailed comparison of the boundary states should be undertaken, using explicit formulas for the matter wavefunctions.

## 9. Relation to matrix models

Finally we come to discuss the connection to the dual matrix model. Clearly, the matrix model description emerges from the Riemann surface $\mathcal{M}_{p,q}$. The FZZT brane corresponds to the macroscopic loop operator of the matrix model, with the $x$ and $y$ the matrix eigenvalue and the resolvent, respectively. The analytic structure of $\mathcal{M}_{p,q}$ dictates the critical behavior of the matrix model, which is the starting point of the double-scaling limit.

It is known that there are many equivalent matrix model descriptions of the $(p, q)$ minimal string theories. The most natural descriptions for our purposes are Kostov's loop gas formalism [30-37] and the two-matrix model [38]. We will focus mainly on the latter. However for actual calculations, especially in the conformal background, the former is often more useful. The two-matrix model consists of two random $N \times N$ hermitian matrices $X$ and $Y$, described by the partition function

$$Z_{\text{matrix}} = \int dX dY \, \exp\left(-\frac{N}{g}\text{Tr}\left(V_1(X) + V_2(Y) - XY\right)\right) . \tag{9.1}$$

In the planar $N \to \infty$ limit, the eigenvalues of $X$ and $Y$ are described by a continuous distribution which can be determined from the structure of the surface $\mathcal{M}_{p,q}$. For instance,

when $p = 2$, the surface can be described as a two-sheeted cover of the complex plane. Then the eigenvalues are localized to the branch cuts of the surface.

Important observables in the two-matrix model are the two resolvents, which are derivatives of the loop operators:

$$R(x) \equiv W'(x) = \mathrm{Tr}\frac{1}{X-x}, \qquad \tilde{R}(y) \equiv \tilde{W}'(y) = \mathrm{Tr}\frac{1}{Y-y} \ . \tag{9.2}$$

These resolvents were calculated in the equivalent loop-gas formalism [30-35] for the $(p,q)$ minimal string theories in the conformal background, and at least for the bosonic models where $(p,q)$ are relatively prime, they are given by the following expressions:

$$\begin{aligned} R(x) &= \left(x+\sqrt{x^2-1}\right)^{q/p} + \left(x-\sqrt{x^2-1}\right)^{q/p} \\ \tilde{R}(y) &= \left(y+\sqrt{y^2-1}\right)^{p/q} + \left(y-\sqrt{y^2-1}\right)^{p/q} \ . \end{aligned} \tag{9.3}$$

One can show that the two resolvents are inverses of one another; in fact this is a consequence of the saddle-point equations of the two-matrix model. Thus we should identify $y = R(x)$ and $x = \tilde{R}(y)$. Moreover, if we write $x = \cosh\theta$ and $y = \cosh\phi$, the equations above become simply the equation $T_p(y) - T_q(x) = 0$ of our surface $\mathcal{M}_{p,q}$. Therefore, the eigenvalues $x$ and $y$ are the boundary cosmological constant and resolvent of the FZZT brane, respectively. Since $y = R(x)$ and $x = \tilde{R}(y)$, we also identify the FZZT brane and the dual brane with the two macroscopic loops built out of $X$ and $Y$, respectively. The fact that there are only two macroscopic loops in the two-matrix model also agrees well with our results in section 3, where we found that there was only one type of FZZT brane (and its dual), labelled by matter state $(1,1)$.

The picture is analogous for the superstring. For $(p,q)$ odd, the matrix model is the same as in the bosonic string, and in particular the expressions for the resolvents are unchanged. This agrees with the fact that we find only one surface in the $(p,q)$ odd superstring, and moreover that it is the same surface as in the bosonic string. For $(p,q)$ even, there are two surfaces, so the two-matrix model description is more difficult. In particular, for a given sign of $\mu$, we can make only one of the surfaces starting from the resolvents (9.2). Perhaps the other surface arises from a more complicated resolvent.

Identifying the coordinates $(x,y)$ of $\mathcal{M}_{p,q}$ with the eigenvalues of $X$ and $Y$ also suggests a way to go beyond tree-level in the minimal string theory. In section 4.1, we discussed the sense in which $x$ and $y$ are conjugate variables. This has a natural interpretation in

the two-matrix model, where in the double-scaling limit the two matrices become the pseudo-differential operators $Q$ and $P$ which satisfy the equation [8]:

$$[Q,P]=\hbar\ . \tag{9.4}$$

This suggests that in order to quantize the minimal string theory, we should promote $x$ and $y$ to operators and quantize the Riemann surface $\mathcal{M}_{p,q}$ [39,40]. This is a promising avenue of investigation that we will leave for future work.

Another connection with the matrix model that we should mention is the relation between matrix model operators and the operators of minimal string theory in the conformal background ($\mu\neq 0$). The natural basis of operators in the matrix model are simply products of $X$ and $Y$; while the tachyons, ground ring elements, etc. are natural to use in minimal string theory. The relations (2.30) and (2.32) in terms of Chebyshev polynomials (and their counterparts in the superstring) tell us how to transform from one basis to another. In [25], this change of basis was worked out in detail for the $(p,q)=(2,2m-1)$ minimal string theories, and in appendix B we check explicitly that our results are in complete agreement. Our results on the $\mu$-deformed ring elements, tachyons and macroscopic loops generalize the work of [25] to all $(p,q)$.

The most detailed picture we have is for theories with $p=2$ which can be described by a one-matrix model. Here we have a unified description of the bosonic models ($q$ odd), the supersymmetric models ($q=4k$) and its generalizations ($q=4k+2$). For the bosonic models, the description is in terms of a one-matrix model with one cut. The curve is

$$2y^2=T_q(x)+1=\frac{\left(T_{\frac{q+1}{2}}(x)+T_{\frac{q-1}{2}}(x)\right)^2}{x+1}\ , \tag{9.5}$$

and the effective eigenvalue potential $V_{eff}(x)$ is obtained by integrating $y$ with respect to $x$ [24]:

$$\sqrt{2}V_{eff}(x)=\frac{T_{\frac{q+3}{2}}(x)+T_{\frac{q+1}{2}}(x)}{(q+2)\sqrt{x+1}}-\frac{T_{\frac{q-1}{2}}(x)+T_{\frac{q-3}{2}}(x)}{(q-2)\sqrt{x+1}}\ . \tag{9.6}$$

On the other hand, the supersymmetric 0B models are represented by a one-matrix model, which has two cuts for $\mu>0$ and no cuts for $\mu<0$ [66-68,24]. Here we have the curve

$$2y^2=\widehat{\eta}\,T_q(x)+1 \tag{9.7}$$

for the $\widehat{\eta}=\pm1$ brane. For the $\widehat{\eta}=-1$ brane, we find

$$V_{eff}(x)\big|_{\widehat{\eta}=-1}=\frac{\left(2q\,T_{\frac{q}{2}}(x)-x\,U_{\frac{q}{2}-1}(x)\right)\sqrt{1-x^2}}{q^2-4}\ , \tag{9.8}$$

while for the $\widehat{\eta} = +1$ brane the result is

$$V_{eff}(x)\big|_{\widehat{\eta}=+1} = \frac{T_{\frac{q}{2}+1}(x)}{q+2} - \frac{T_{\frac{q}{2}-1}(x)}{q-2} \ . \tag{9.9}$$

For example in pure supergravity, $(p,q) = (2,4)$ and the curve for the $\widehat{\eta} = -1$ brane becomes $y^2 = -4x^2(x^2-1)$, while for $\widehat{\eta} = 1$ the curve is $y^2 = (2x^2-1)^2$. In general, we find that the curve has no cuts for $\widehat{\eta} = +1$ (it is polynomial in $x$) and has two cuts starting at $x = \pm 1$ for $\widehat{\eta} = -1$. Thus the resolvent of the 0B matrix model in the positive (negative) $\mu$ phase corresponds to the $\widehat{\eta} = -1$ $(+1)$ brane. Evidently, only the $\eta = -1$ brane can be interpreted as the resolvent in the 0B matrix model.

Using a change of variables, we can also obtain the resolvents of the 0A matrix models for the $(2, 2k)$ theories. The change of variables is motivated by the fact that in 0A, the natural variable is not the eigenvalue $x_B$, but rather $x_A = x_B^2$. If we think of the resolvents for 0A and 0B as a one-forms on the Riemann surface, imposing that the one-forms transform into one another results in $y_B\, dx_B = y_A\, dx_A = 2y_A x_B\, dx_B$, so we find

$$x_A = x_B^2, \qquad y_A = \frac{y_B}{2x_B} \ . \tag{9.10}$$

Thus the resolvent of 0A can actually have a pole at $x_A = 0$, a fact noticed in [24]. There it was convenient to define the curve of 0A in terms of

$$\widehat{y}_A \equiv 2x_A y_A = x_B y_B \ . \tag{9.11}$$

Using (9.10) and (9.11), we find the 0A curves:

$$2\widehat{y}_A^2 = x_A \left( \widehat{\eta}\, T_{\frac{q}{2}}(2x_A - 1) + 1 \right) \ . \tag{9.12}$$

Again, for pure supergravity we recognize the resolvent of the 0A matrix model in the two phases as the $\eta = -1$ brane. For positive $\mu$ ($\widehat{\eta} = -1$) it satisfies $\widehat{y}_A^2 = 4x_A^2(1 - x_A)$ and for negative $\mu$ ($\widehat{\eta} = +1$) it is $\widehat{y}_A^2 = x_A(2x_A - 1)^2$. Moreover, at least for pure supergravity, the $\eta = \pm 1$ brane in 0A is the same as the $\eta = \mp 1$ brane in 0B. As we have already seen, the $\eta = +1$ 0B brane cannot be described easily in the 0B language. Instead, it has a natural description as the resolvent of the 0A theory.

The description in terms of the effective potential gives us a more physical understanding of the ZZ branes in the one-matrix model. The singularities of the surface correspond to points where $y(x) = T_q'(x) = 0$, i.e. the regular zeros of $y$. In terms of the effective

eigenvalue potential $V_{eff}(x)$, this is the statement that the singularities are local extrema of $V_{eff}(x)$. Therefore the ZZ branes are matrix eigenvalues located at the zero-force points of the effective potential. In order to create a ZZ brane, we must pull it out of the Fermi sea, which corresponds to the branch cut of our surface. This is the meaning of the ZZ brane as a contour integral: the integral pulls an eigenvalue from the Fermi sea onto the singularity (zero-force point).

Note that the extrema of $V_{eff}(x)$ are not necessarily local maxima. Let us focus on the bosonic models for concreteness, whose effective potential is given by (9.6). It is easy to show that the principal $(1, n)$ ZZ branes with $n$ even (odd) lie at local minima (maxima) of $V_{eff}$, and moreover at these extrema, $V_{eff}$ takes the values

$$V_{eff}(x_n) \sim (-1)^{n+1} \sin \frac{2\pi n}{2m-1} \ . \tag{9.13}$$

Recall that the "Fermi level" of the perturbative vacuum is located at $V_{eff} = 0$. Then according to (9.13), all the minima of $V_{eff}$ lie *below the Fermi sea*. Apparently, all of these models are slightly unstable to the tunneling of a small number of eigenvalues into the minima of $V_{eff}$.

This detailed discussion has so far been only for the one-matrix model. The picture in the two matrix model is not nearly as complete. For example, it is not clear how to obtain the 0A curves. Presumably, this can be done by considering a two-matrix model of complex matrices. Also, a description in terms of an effective potential is lacking, although some interesting proposals were advanced in [69]. In any event, we still expect that in some sense the singularities of $\mathcal{M}_{p,q}$ are "zero-force" points of some effective potential.

The effect of adding order $1/g_s$ ZZ branes on the Riemann surface has a natural interpretation in the matrix model. As in [55,56], the contour integral (4.32)(7.18)

$$\oint_{A_{m,n}} y dx = g_s N_{m,n} \ . \tag{9.14}$$

measures the number of eigenvalues around $(x_{mn}, y_{mn})$. We again see that the matrix model eigenvalues can be thought of as ZZ branes with the different $(m, n)$ ZZ brane differ by their position in the surface.

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## Appendix A. The Backlund transformation and FZZT branes

The purpose of this appendix is to give a semiclassical, intuitive picture of the FZZT branes using the Backlund transformation.

### *A.1. Bosonic Liouville theory*

We start with the bosonic Liouville theory in a two-dimensional, Lorentzian signature spacetime (our conventions are chosen to agree with [14]):

$$\mathcal{L} = \frac{1}{4\pi}\left((\partial_\tau\phi)^2 - (\partial_\sigma\phi)^2 - 4\pi\mu e^{2b\phi}\right) \ . \tag{A.1}$$

The most general classical solution to the equations of motion is given in terms of two arbitrary functions $A^\pm(x^\pm)$

$$e^{2b\phi_{cl}} = \frac{1}{\pi b^2\mu}\frac{\partial_+ A^+\partial_- A^-}{(A^+ - A^-)^2} \ , \tag{A.2}$$

where $x^\pm = \tau \pm \sigma$ and $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. The Backlund field is a free field which is defined in terms of $A^\pm$

$$\tilde{\phi} = \frac{1}{2b}\log\left(\frac{\partial_+ A^+}{\partial_- A^-}\right) \ . \tag{A.3}$$

Eliminating $A^\pm$ we find

$$\begin{aligned}\partial_\sigma\phi &= \partial_\tau\tilde{\phi} - \sqrt{4\pi\mu}e^{b\phi}\cosh(b\tilde{\phi})\\ \partial_\tau\phi &= \partial_\sigma\tilde{\phi} - \sqrt{4\pi\mu}e^{b\phi}\sinh(b\tilde{\phi}) \ ,\end{aligned} \tag{A.4}$$

or equivalently

$$\pm\partial_\pm\phi = \partial_\pm\tilde{\phi} - \frac{\sqrt{4\pi\mu}}{2}e^{b(\phi\pm\tilde{\phi})} \ . \tag{A.5}$$

The generating functional of this transformation is

$$\int d\sigma \left( \phi \partial_\sigma \tilde{\phi} - \frac{\sqrt{4\pi\mu}}{b} e^{b\phi} \sinh(b\tilde{\phi}) \right) \ . \tag{A.6}$$

Alternatively, we can start from the transformation (A.4) or (A.5) and check that they are compatible only if the appropriate equations of motion are satisfied

$$\partial_+ \partial_- \phi = -\pi b \mu e^{2b\phi} \ , \qquad \partial_+ \partial_- \tilde{\phi} = 0 \ . \tag{A.7}$$

It is straightforward to evaluate the energy momentum tensor

$$T_{\pm\pm} = (\partial_\pm \phi)^2 - \frac{1}{b} \partial_\pm^2 \phi = (\partial_\pm \tilde{\phi})^2 \mp \frac{1}{b} \partial_\pm^2 \tilde{\phi} \ , \tag{A.8}$$

where the second term is the improvement term (note that it is different for $\phi$ and $\tilde{\phi}$). Using the classical solution (A.2) and the energy momentum tensor (A.8), one can check that

$$\left[\partial_+^2 - b^2 T_{++}(\phi_{cl})\right] \frac{1}{\sqrt{\partial_+ A^+}} = \left[\partial_+^2 - b^2 T_{++}(\phi_{cl})\right] \frac{A^+}{\sqrt{\partial_+ A^+}} = 0 \ , \tag{A.9}$$

and therefore

$$\left[\partial_+^2 - b^2 T_{++}(\phi_{cl})\right] e^{-b\phi_{cl}} = 0 \ . \tag{A.10}$$

In the quantum theory this is the statement that $e^{-b\phi}$ has a null descendant. This null vector is the basis for the exact solution of the quantum theory [11-13].

The first term in the transformation (A.5) shows that it is like a T-duality transformation. Therefore, we expect that Dirichlet and Neumann boundary conditions are exchanged. Indeed, the FZZT brane which is associated with the Neumann boundary conditions

$$\partial_\sigma \phi = -2\pi b \mu_B e^{b\phi} \tag{A.11}$$

becomes a Dirichlet brane in terms of $\tilde{\phi}$ with

$$\frac{\mu_B}{\sqrt{\mu}} = \frac{1}{b\sqrt{\pi}} \cosh(b\tilde{\phi}) \ . \tag{A.12}$$

Note that (A.12) is the semiclassical ($b \to 0$) limit of (3.3), before the rescaling of $\mu$ and $\mu_B$. Therefore we identify the Backlund field with the parameter $\pi\sigma$ of the FZZT brane.

The fact that Dirichlet and Neumann boundary conditions are exchanged by the Backlund transformation is also clear from the form of the energy momentum tensor $T_{\pm\pm}$ of (A.8). Because of the improvement term Dirichlet boundary conditions of $\phi$ are not conformally invariant. However, in terms of $\tilde{\phi}$ the improvement terms in $T_{++}$ and $T_{--}$ have opposite values. Therefore, Neumann boundary conditions for $\tilde{\phi}$ are not conformally invariant but Dirichlet boundary conditions are consistent. Equivalently, a would-be localized D-brane at $\phi$ is expected to have mass proportional to $e^{\frac{\phi}{b}}$. Therefore, it is unstable and is pushed to $\phi \to +\infty$. On the other hand a D-brane localized at $\tilde{\phi}$ is stable.

*A.2. Minisuperspace wavefunctions*

In the minisuperspace approximation we focus on the zero mode of $\phi$. We can take as a complete basis of states the eigestates of $\phi$ or the states with energy $\frac{1}{2}P^2$. Their inner products are

$$\begin{aligned}
\langle\phi_1|\phi_2\rangle &= \delta(\phi_1-\phi_2)\\
\langle P_1|P_2\rangle &= \pi\delta(P_1-P_2)\\
\langle\phi|P\rangle &= \frac{2\left(\frac{\pi\mu}{b^2}\right)^{-\frac{iP}{b}}}{\Gamma\left(-\frac{2iP}{b}\right)}K_{\frac{2iP}{b}}\left(\frac{\sqrt{4\pi\mu}}{b}e^{b\phi}\right)\\
&= e^{2iP\phi}(1+\ldots)+\left(\frac{\pi\mu}{b^2}\right)^{-\frac{2iP}{b}}\frac{\Gamma\left(\frac{2iP}{b}\right)}{\Gamma\left(-\frac{2iP}{b}\right)}e^{-2iP\phi}(1+\ldots)\ .
\end{aligned} \tag{A.13}$$

The states $|P\rangle$ were normalized such that the incoming wave $e^{2iP\phi}$ in $\langle\phi|P\rangle$ has weight one. The coefficient in front of $e^{-2iP\phi}$ is a pure phase and is the reflection amplitude.

The boundary states are eigenstates of $\tilde\phi$. They satisfy

$$\begin{aligned}
\langle\phi|\tilde\phi\rangle &= e^{-2\pi\mu_B e^{b\phi}}, \qquad\qquad \mu_B=\frac{\sqrt{\mu}}{b\sqrt{\pi}}\cosh(b\tilde\phi)\\
\langle\tilde\phi_1|\tilde\phi_2\rangle &= -\frac{1}{b}\log\left[\frac{\sqrt{4\pi\mu}}{b}\left(\cosh(b\tilde\phi_1)+\cosh(b\tilde\phi_2)\right)\right]+const\\
&= -\frac{1}{b}\log\left(\frac{2\sqrt{4\pi\mu}}{b}\cosh\frac{b(\tilde\phi_1+\tilde\phi_2)}{2}\cosh\frac{b(\tilde\phi_1-\tilde\phi_2)}{2}\right)+const\\
\langle\tilde\phi|p\rangle &= \frac{2}{b}\Gamma\left(\frac{2iP}{b}\right)\left(\frac{\pi\mu}{b^2}\right)^{-i\frac{P}{b}}\cos(P\tilde\phi)\ .
\end{aligned} \tag{A.14}$$

The additive constant in $\langle\tilde\phi_1|\tilde\phi_2\rangle$ is a nonuniversal infinite constant which is independent of $\tilde\phi_{1,2}$. The wave function $\langle\phi|\tilde\phi\rangle$ can be derived using the generating functional (A.6) after exchanging the role of $\tau$ and $\sigma$. Alternatively, we can derive it in a Euclidean worldsheet. Note that the states $|\tilde\phi\rangle$ are not orthonormal. This means that the canonical transformation from $\phi$ to $\tilde\phi$ is not unitary. This fact is also the reason for the somewhat unusual decomposition

$$|\tilde\phi\rangle=\frac{2}{b\pi}\int_0^\infty dP\ \cos(2P\tilde\phi)\Gamma\left(-\frac{2iP}{b}\right)\left(\frac{\pi\mu}{b^2}\right)^{\frac{iP}{b}}|P\rangle\ . \tag{A.15}$$

The fact that is the semiclassical limit of the Liouville part of the FZZT boundary state (3.1) confirms our identification of the Backlund field with the parameter $\pi\sigma$ of the FZZT brane. It is interesting that if not for the factor of $\Gamma\left(-\frac{2ip}{b}\right)\left(\frac{\pi\mu}{b^2}\right)^{i\frac{p}{b}}$, this would have meant that $\tilde\phi$ is conjugate to $p$ and $|\tilde\phi\rangle$ is a standard position eigenstate. However, since this factor depends only on $p$ and not on $\tilde\phi$, that conclusion is not completely wrong.

*A.3. Supersymmetric Liouville theory*

Now let us consider super-Liouville theory with a Euclidean worldsheet. We follow the conventions of [23], except that here we rescale $\mu \to \mu/2$. The covariant derivatives, supercharges and algebra are

$$\begin{aligned} D &= \frac{\partial}{\partial\theta} + \theta\partial \quad , \qquad \overline{D} = \frac{\partial}{\partial\overline{\theta}} + \overline{\theta}\overline{\partial} \quad , \qquad \{D, D\} = 2\partial \quad , \qquad \{\overline{D}, \overline{D}\} = 2\overline{\partial} \\ Q &= \frac{\partial}{\partial\theta} - \theta\partial \quad , \qquad \overline{Q} = \frac{\partial}{\partial\overline{\theta}} - \overline{\theta}\overline{\partial} \quad , \qquad \{Q, Q\} = -2\partial \ , \qquad \{\overline{Q}, \overline{Q}\} = -2\overline{\partial} \end{aligned} \tag{A.16}$$

with all other (anti)commutators vanishing. We define $z = x + iy$, $\overline{z} = x - iy$ and therefore $\partial = (\partial_x - i\partial_y)/2$, $\overline{\partial} = (\partial_x + i\partial_y)/2$. Finally, the integration measure is $\int d^2z d^2\theta = 2\int dxdyd\overline{\theta}d\theta$. The action for

$$\Phi = \phi + i\theta\psi + i\overline{\theta}\overline{\psi} + i\theta\overline{\theta}F \tag{A.17}$$

is the super-Liouville action

$$S = \frac{1}{4\pi}\int d^2zd^2\theta \left[D\Phi\overline{D}\Phi + i\mu e^{b\Phi}\right] . \tag{A.18}$$

The Backlund transformation in super-Liouville theory [70,71] is

$$\begin{aligned} D\Phi &= D\tilde{\Phi} + \xi b\sqrt{|\mu|}\Gamma e^{\frac{b}{2}(\Phi+\tilde{\Phi})} \\ \overline{D}\Phi &= -\overline{D}\tilde{\Phi} - \xi\zeta b\sqrt{|\mu|}\Gamma e^{\frac{b}{2}(\Phi-\tilde{\Phi})} \\ D\Gamma &= -i\xi\sqrt{\frac{|\mu|}{4}}e^{\frac{b}{2}(\Phi+\tilde{\Phi})} \\ \overline{D}\Gamma &= -i\xi\zeta\sqrt{\frac{|\mu|}{4}}e^{\frac{b}{2}(\Phi-\tilde{\Phi})} \\ \zeta &= \mathrm{sign}(\mu) \ , \end{aligned} \tag{A.19}$$

where $\Gamma$ is a fermionic superfield ($\Gamma^2 = 0$), and $\tilde{\Phi}$ is the Backlund field. (We have rescaled the variables to agree with the conventions of [23].) The parameter $\xi = \pm 1$ implements the $(-1)^{F_L}$ symmetry, as discussed in section 6.1. It is straightforward to check that the integrability conditions for (A.19) are the equations of motion

$$\begin{aligned} &\overline{D}D\Phi + \frac{i\mu b}{2}e^{b\Phi} = 0 \\ &\overline{D}D\tilde{\Phi} = 0 \\ &\overline{D}D\Gamma - \frac{i\mu b^2}{8}e^{b\Phi}\Gamma = 0 \ , \end{aligned} \tag{A.20}$$

i.e. $\Phi$ satisfies the equation of motion of (A.18), and $\tilde{\Phi}$ is free.

The energy momentum tensor superfield is

$$\begin{aligned} T &= D\Phi D^2\Phi - \frac{1}{b}D^3\Phi = D\tilde{\Phi}D^2\tilde{\Phi} - \frac{1}{b}D^3\tilde{\Phi} \\ \overline{T} &= \overline{D}\Phi\overline{D}^2\Phi - \frac{1}{b}\overline{D}^3\Phi = \overline{D}\tilde{\Phi}\overline{D}^2\tilde{\Phi} + \frac{1}{b}\overline{D}^3\tilde{\Phi} \ . \end{aligned} \tag{A.21}$$

The second term is the improvement term. It has the same sign in $T$ and $\overline{T}$ when expressed in terms of $\Phi$, but it has opposite signs in $T$ and $\overline{T}$ when expressed in terms of $\tilde{\Phi}$. The fermionic superfield $\Gamma$ does not contribute to the energy momentum tensor.

The FZZT branes are D0-branes of $\tilde{\Phi}$; i.e. the Backlund field satisfies Dirichlet boundary conditions

$$D_t\tilde{\Phi} = (D + \eta\overline{D})\tilde{\Phi} = 0 \ , \tag{A.22}$$

where $\eta = \pm 1$ denotes the preserved subspace of superspace: $\theta = \eta\overline{\theta}$. The opposite signs of the improvement term in $T$ and $\overline{T}$ when expressed in terms of $\tilde{\Phi}$ (A.21), make these boundary conditions conformal (conversely, Neumann boundary conditions on $\tilde{\Phi}$ are not consistent). From (A.19) we find the boundary conditions of the other fields

$$\begin{aligned} D_n\Phi &= (D - \eta\overline{D})\Phi = 2b\mu_B\Gamma e^{\frac{b}{2}\Phi} \\ D_t\Gamma &= (D + \eta\overline{D})\Gamma = -i\mu_B e^{\frac{b}{2}\Phi} \ , \end{aligned} \tag{A.23}$$

where

$$\mu_B = \begin{cases} \xi\sqrt{|\mu|}\cosh(\frac{b}{2}\tilde{\Phi}) & \widehat{\eta} = +1 \\ \xi\sqrt{|\mu|}\sinh(\frac{b}{2}\tilde{\Phi}) & \widehat{\eta} = -1 \end{cases} \tag{A.24}$$

with $\widehat{\eta} = \zeta\eta$. This expression for $\mu_B$ is the semiclassical limit of (6.13). Thus, as in the bosonic string, we are led to identify the Backlund field with the parameter $\sigma$ of the FZZT brane. Note also that the equations (A.23) are the boundary equations of motion when we add to the action (A.18) the boundary term

$$\begin{aligned} S_{\text{bry}} &= \frac{1}{2\pi}\oint dx d\theta_t \left(\Gamma D_t\Gamma + 2i\mu_B\Gamma e^{\frac{b}{2}\Phi}\right) \\ &= \frac{1}{2\pi}\oint dx\Big[-\gamma\partial_x\gamma - f^2 - \mu_B\Big(b\gamma(\psi + \eta\overline{\psi})e^{\frac{b}{2}\phi} + 2fe^{\frac{b}{2}\phi}\Big)\Big] \\ &= \frac{1}{2\pi}\oint dx\Big[-\gamma\partial_x\gamma - \mu_B\, b\,\gamma(\psi + \eta\overline{\psi})e^{\frac{b}{2}\phi} + \mu_B^2\, e^{b\phi}\Big] \ , \end{aligned} \tag{A.25}$$

where $\Gamma = \gamma + i\theta_t f$ is a fermionic superfield at the boundary.

*A.4. Minisuperspace wavefunctions*

We denote by $|\phi\pm\rangle$ and $|P\pm\rangle$ the states with the two different fermion number in the R sector and by $|\phi 0\rangle$ and $|P0\rangle$ the states in the NS sectors. Following [23] the wave functions are expressed in terms of $z = |\mu|e^{b\phi}$. Let us assume first that $\mu$ is positive ($\zeta = \mathrm{sign}(\mu) = +1$). Then the wavefunctions are[9]

$$\begin{aligned}
&\langle\phi_1 \pm |\phi_2 \pm\rangle = \delta(\phi_1 - \phi_2)\\
&\langle\phi_1 0|\phi_2 0\rangle = \delta(\phi_1 - \phi_2)\\
&\langle P_1 \pm |P_2 \pm\rangle = 2\pi\delta(P_1 - P_2)\\
&\langle P_1 0|P_2 0\rangle = 2\pi\delta(P_1 - P_2)\\
&\Psi_{P\pm}(\phi) = \langle\phi \pm |P\pm\rangle = \frac{2}{\Gamma\left(-\frac{iP}{b}+\frac{1}{2}\right)}\left(\frac{|\mu|}{4}\right)^{-\frac{iP}{b}}\sqrt{z}\left(K_{\frac{iP}{b}-\frac{1}{2}}(z) \pm K_{\frac{iP}{b}+\frac{1}{2}}(z)\right)\\
&\qquad = e^{iP\phi}\left(1+\mathcal{O}(z)\right) \pm \frac{\Gamma\left(\frac{iP}{b}+\frac{1}{2}\right)}{\Gamma\left(-\frac{iP}{b}+\frac{1}{2}\right)}\left(\frac{|\mu|}{4}\right)^{-\frac{2iP}{b}} e^{-iP\phi}\left(1+\mathcal{O}(z)\right)\\
&\Psi_{P0} = \langle\phi 0|P0\rangle = \frac{2}{\Gamma\left(-\frac{iP}{b}\right)}\left(\frac{|\mu|}{4}\right)^{-\frac{iP}{b}} K_{\frac{iP}{b}}(z)\\
&\qquad = e^{iP\phi}\left(1+\mathcal{O}(z)\right) - \frac{\Gamma(1+\frac{iP}{b})}{\Gamma\left(1-\frac{iP}{b}\right)}\left(\frac{|\mu|}{4}\right)^{-\frac{2iP}{b}} e^{-iP\phi}\left(1+\mathcal{O}(z)\right)\ .
\end{aligned} \tag{A.26}$$

The wavefunctions satisfy

$$\begin{aligned}
&(z\partial_z \pm z)\Psi_{P\pm}(z) = \frac{iP}{b}\Psi_{P\mp}(z)\\
&\left(-(z\partial_z)^2 + z^2 - \frac{P^2}{b^2}\right)\Psi_{P0}(z) = 0\ .
\end{aligned} \tag{A.27}$$

When $\mu$ changes sign $z$ which is defined in terms of $|\mu|$ does not change. Therefore the equations (A.27) are unchanged. However, since these equations are derived from the action of the supercharges, the term linear in $z$ must change sign. This means that the wavefunctions $\Psi_{P\pm}(z)$ have the same functional form when $\mu$ changes sign but they occur for the states with opposite fermion number; i.e.

$$\langle\phi \pm |P\pm\rangle = \Psi_{P,\pm\zeta}(z)\ . \tag{A.28}$$

[9] We reversed the $\pm$ label of $\Psi_{P\pm}$ and changed the normalization of the wavefunctions relative to [23] to have the coefficient of $e^{iP\phi}$ in the wavefunctions normalized to one.

In order to study D-branes we need eigenstates of $\tilde{\phi}$. In the R sector they are $|\tilde{\phi}\pm\rangle$ and in the NS sector $|\tilde{\phi}0\rangle$. After integrating out the various fermions as in [23], we have

$$\begin{aligned}
&\langle\phi 0|\tilde{\phi}0\rangle = e^{-z\cosh(b\tilde{\phi})}\\
&\langle\phi\pm|\tilde{\phi}\pm\rangle = \sqrt{z}e^{-z\cosh(b\tilde{\phi})}\left(e^{\frac{b\tilde{\phi}}{2}}\pm\zeta e^{-\frac{b\tilde{\phi}}{2}}\right)\\
&\langle\tilde{\phi}0|P0\rangle = \frac{2}{b}\left(\frac{|\mu|}{4}\right)^{-\frac{iP}{b}}\Gamma\left(\frac{iP}{b}\right)\cos(P\tilde{\phi})\\
&\langle\tilde{\phi}\pm|P\pm\rangle = \frac{2}{b}\left(\frac{|\mu|}{4}\right)^{-\frac{iP}{b}}\Gamma\left(\frac{iP}{b}+\frac{1}{2}\right)\left(e^{-iP\tilde{\phi}}\pm\zeta e^{iP\tilde{\phi}}\right)\ .
\end{aligned}\tag{A.29}$$

From these we derive the decompositions

$$\begin{aligned}
|\tilde{\phi}0\rangle &= \frac{1}{b\pi}\int_0^\infty dP\ \cos(P\tilde{\phi})\Gamma\left(-\frac{iP}{b}\right)\left(\frac{|\mu|}{4}\right)^{\frac{iP}{b}}|P0\rangle\\
|\tilde{\phi}\pm\rangle &= \frac{1}{b\pi}\int_0^\infty dP\ \left(e^{iP\tilde{\phi}}\pm\zeta e^{-iP\tilde{\phi}}\right)\Gamma\left(-\frac{iP}{b}+\frac{1}{2}\right)\left(\frac{|\mu|}{4}\right)^{\frac{iP}{b}}|P\pm\rangle\ .
\end{aligned}\tag{A.30}$$

These formulas for the boundary states agree with the semiclassical limit of (6.12).

## Appendix B. Tests with the one-matrix model

In this appendix, we will study in detail the $(2,2m-1)$ bosonic minimal models, which can be described very simply in terms of the $m$th critical point of a one-matrix model. This will serve as a useful check of the general analysis of the bosonic $(p,q)$ models in sections 3 and 4 and the discussion of the matrix model in section 9. In the one matrix model, there is only one resolvent, so we will focus on the FZZT brane and not its dual. The resolvent is given by (9.5) with $q=2m-1$:

$$y_m = (\sqrt{\mu})^{m-1/2}\frac{T_m(x)+T_{m-1}(x)}{\sqrt{1+x}}\ ,\tag{B.1}$$

where here we have restored the overall power of $\mu$ in front of $y$ for the purposes of the present discussion. Our goal will be to explicitly confirm (B.1) using the one-matrix model. The method will be as follows. Turning on $\mu$ in the minimal string theory corresponds to turning on a specific set of scaling perturbations in the matrix model. Using the techniques of [25], we can perform the change of basis from $\mu$ to scaling perturbations of the matrix model, and then explicitly compute the resolvent at non-zero $\mu$, confirming (B.1).

Exactly at the $m$th critical point of the one-matrix model, the resolvent takes the form $y = (\sqrt{\mu})^{m-1/2} x^{m-1/2}(x-b)^{1/2}$. A general perturbation around the $m$th critical point can be written as

$$y = \sum_{j=1}^{m} t_j y_j \ , \tag{B.2}$$

where the $t_j$ are the couplings to the $j$th matrix model scaling operator, and the $y_j$ are given by (see e.g. section 2.2 of [10]):

$$y_j(x) = \frac{b^{1/2}(\sqrt{\mu})^{j-1/2}}{B(j,\frac{1}{2})} \int_x^a ds\,(s-x)^{-1/2} s^{j-1} \ . \tag{B.3}$$

Here we are using the standard "Gelfand-Dikii" normalization of the couplings $t_j$, so that the string equation takes the form

$$\sum_j t_j R_j[u] = 0 \ . \tag{B.4}$$

Now, in order to compare with the continuum description in the "conformal background," i.e. with a fixed cosmological constant $\mu$, we must express the $t_j$ in terms of $\mu$. Explicit formulas for this change of basis were derived in [25]:

$$\begin{aligned} t_{m-2p} &= \frac{c_{m-2p}}{a_{m-2p}} \mu^p \\ c_{m-2p} &= \frac{(-1)^{m+1}\pi}{\sqrt{8}} \frac{2^{m-2p}}{(m-2p)!\,p\,!\,\Gamma(p-m+\frac{3}{2})} \\ a_{m-2p} &= \frac{(-1)^m}{2^{2(m-2p)}} \binom{2m-4p-1}{m-2p} \ . \end{aligned} \tag{B.5}$$

Here $a_k$ is the leading coefficient of the Gelfand-Dikii polynomial $R_k[u]$, i.e. $R_k[u] = a_k u^k + \ldots$. We must include this factor when using the results of [25], because there a slightly different normalization of the couplings $t_j$ was used, in which the (genus zero) string equation has the form $\sum_i t_j u^j = 0$. After some gamma function manipulations, we obtain the following simplified form of $t_{m-2p}$:

$$t_{m-2p} = \frac{(-1)^{m+1}\sqrt{2\pi}}{4(m-p)^2-1} \frac{2^{m-2p+1}}{B(p+1,-\frac{1}{2}-m+p)} \mu^p \ . \tag{B.6}$$

Substituting our formula for $t_j$ in (B.2), we find the form of the resolvent in the conformal background:

$$\begin{aligned} y(x) = (-1)^{m+1} b^{1/2} (\sqrt{\mu})^{m-1/2} \sqrt{2\pi} \int_x^a ds\,(s-x)^{-1/2} \times \\ \sum_{p=0}^{[\frac{m-1}{2}]} \frac{1}{4(m-p)^2-1} \frac{2^{m-2p+1}}{B(m-2p,\frac{1}{2}) B(p+1,p-m-\frac{1}{2})} s^{m-2p-1} \ . \end{aligned} \tag{B.7}$$

Using the identity

$$\frac{(-1)^{p+1}2^{2m-2p}}{B(m-2p,\frac{1}{2})B(p+1,p-m-\frac{1}{2})} = \left(4(m-p)^2-1\right)\binom{2m-2p-2}{m-1}\binom{m-1}{p} \tag{B.8}$$

in (B.7), we obtain

$$\begin{aligned} y(x) = (-1)^m b^{1/2}(\sqrt{\mu})^{m-1/2}\sqrt{2\pi}\int_x^a ds\,(s-x)^{-1/2}\times \\ \frac{1}{2^{m-1}}\sum_{p=0}^{\left[\frac{m-1}{2}\right]}(-1)^p\binom{2m-2p-2}{m-1}\binom{m-1}{p}s^{m-2p-1}\ . \end{aligned} \tag{B.9}$$

We recognize the sum in (B.9) as an explicit form of the Legendre polynomial $P_{m-1}(x)$. Furthermore, the integral can be evaluated using standard tables assuming $a=-1$. Thus we arrive at the final form of the matrix model curve:

$$\begin{aligned} y(x) &= b^{1/2}(-1)^{m+1}\sqrt{2\pi}(\sqrt{\mu})^{m-1/2}\int_{-1}^x ds\,(s-x)^{-1/2}P_{m-1}(s) \\ &= \frac{(-1)^m\sqrt{8\pi}}{2m-1}ib^{1/2}(\sqrt{\mu})^{m-1/2}\left(\frac{T_m\,(x)+T_{m-1}\,(x)}{\sqrt{1+x}}\right)\ , \end{aligned} \tag{B.10}$$

which is indeed in exact agreement with the continuum prediction (B.1), up to an irrelevant overall normalization that can be absorbed into the definition of $b$.

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